Lesson 14 – Working with Polynomials – FTA, Inequalities, Multiplicity & Roots of Unity

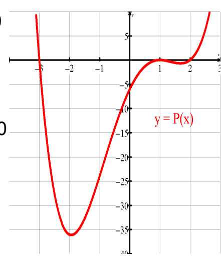
HL1 Math - Santowski

(A) Opening Example - Quartics

- Solve $x^4 - x^3 - 7x^2 + 13x - 6 = 0$
- Hence solve $x^4 - x^3 + 13x \geq 6 + 7x^2$
- Hence, sketch a graph of y = | P(|x|) |
- Hence, sketch a graph of the reciprocal of
  $y = x^4 - x^3 - 7x^2 + 13x - 6$

(A) Opening Example - Solutions

- Solve $x^4 – x^3 – 7x^2 + 13x – 6 = 0$
- $P(x) = 0 = (x – 1)^2(x + 3)(x – 2)$
- Solve $x^4 – x^3 – 7x^2 + 13x – 6 \geq 0$
- So $P(x) \geq 0$ on $[-\infty,-3)$ or $(2, \infty)$



(B) Multiplicity of Roots

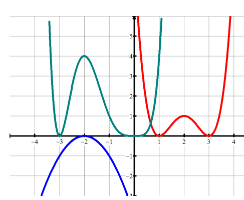
- Factor the following polynomials:
- (i) $P(x) = x^2 – 14x + 49$
- (ii) $P(x) = x^3 + 3x^2 + 3x + 1$
- (iii) Now solve each polynomial equation, P(x) = 0
- (iv) Solve $0 = 5(x + 1)^2(x – 2)^3$
- (v) Solve $0 = x^4(x – 3)^2(x + 5)$
- (vi) Solve $0 = (x + 1)^3(x – 1)^2(x – 5)(x + 4)$
- Graph each polynomial from Q iv, v, vi

(B) Multiplicity of Roots

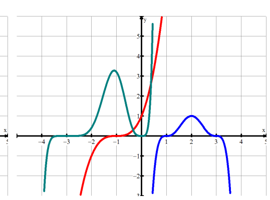
- If *r* is a zero of a polynomial and the exponent on the factor that produced the root is *k*, $(x – r)^k$, then we say that *r* has **multiplicity** of *k*. Zeroes with a multiplicity of 1 are often called **simple** zeroes.
- For example, the polynomial $x^2 – 14x + 49$ will have one zero, x = 7, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as, $(x – 7)^2 = (x – 7)(x – 7)$
- Written this way the term (x – 7) shows up twice and each term gives the same zero, x = 7.
- Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

(B) Multiplicity ➔ Graphic Connection

Even Multiplicity



Odd Multiplicity



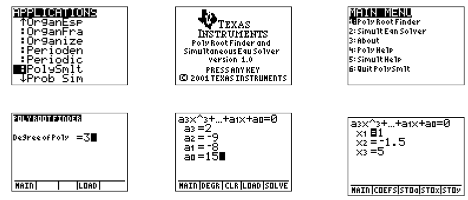
(C) Solving & Factoring on the TI-84

- Use your calculator to answer the following Qs:
- (a) Solve $2x^3 - 9x^2 - 8x = -15$ non-graphically
- (b) Solve $0 = 2x^3 - 9x^2 + 7x + 6$ non-graphically
- (c) Use a graph to factor

  $P(x) = x^4 - x^3 - 7x^2 + 13x - 6$

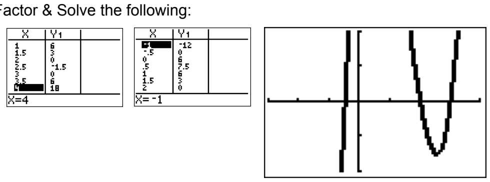
(C) Solving & Factoring on the TI-84

- Solve $2x^3 - 9x^2 - 8x = -15$ turn it into a "root" question ➔ i.e Solve $P(x) = 0$ ➔ Solve $0 = 2x^3 - 9x^2 - 8x + 15$



(C) Solving & Factoring on the TI-84

- Factor & Solve the following:



- $0 = 2x^3 - 9x^2 + 7x + 6$ ➔ roots at $x = -0.5, 2, 3$ ➔ would imply factors of $(x - 2)$, $(x - 3)$ and $(x + ½)$ ➔ $P(x) = 2(x + ½)(x - 2)(x - 3)$
- So when factored $P(x) = (2x + 1)(x - 2)(x - 3)$

(C) Solving & Factoring on the TI-84

- (CA) Factor the following polynomials:
- $P(x) = -2x^3 - x^2 + 25x - 12$
- $P(x) = 4x^3 - 12x^2 - 19x + 12$
- $P(x) = 12x^4 + 32x^3 - 15x^2 - 8x + 3$

(D) Using Complex Numbers ➔ Solving Equations

- One real root of a cubic polynomial -4 and one complex root is $2i$
- (a) What is the other root?
- (b) What are the factors of the cubic?
- (c) If the y-intercept of the cubic is -4, determine the equation in factored form and in expanded form.

(D) Solving Polynomials if $x \in$ **C**

- Factor and solve $3 - 2x^2 - x^4 = 0$ if $x \in$ **C**
- Factor and solve $3x^3 - 7x^2 + 8x - 2 = 0$ if $x \in$ **C**
- Factor and solve $2x^3 + 14x - 20 = 9x^2 - 5$ if $x \in$ **C**
- Now write each polynomial as a product of its factors
- Explain the graphic significance of your solutions for x

(D) Solving if x ε **C** – Solution to Ex 1

- Factor and solve $3 - 2x^2 - x^4 = 0$ if x ε **C** and then write each polynomial as a product of its factors
- Solutions are $x = \underline{+}1$ and $x = \underline{+}i\sqrt{3}$
- So rewriting the polynomial in factored form (over the reals) is $P(x) = -(x^2 + 3)(x - 1)(x + 1)$
- And in factored form over the complex numbers:

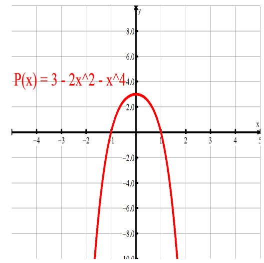
$$P(x) = -(x-1)(x+1)\left(x - i\sqrt{3}\right)\left(x + i\sqrt{3}\right)$$

(D) Solving if x ε **C** – Graphic Connection

- With $P(x) = 3 - 2x^2 - x^4$, we can now consider a graphic connection, given that $P(x) = -(x^2 + 3)(x - 1)(x + 1)$

  or given that

$$P(x) = -(x-1)(x+1)\left(x - i\sqrt{3}\right)\left(x + i\sqrt{3}\right)$$

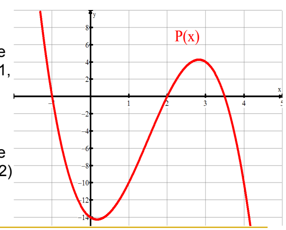


(E) Fundamental Theorem of Algebra

- The fundamental theorem of algebra can be stated in many ways:
  - (a) If P(x) is a polynomial of degree *n* then P(x) will have exactly *n* zeroes (real or complex), some of which may repeat.
  - (b) Every polynomial function of degree $n \underline{>} 1$ has exactly *n* complex zeroes, counting multiplicities
  - (c) If P(x) has a nonreal root, $a+bi$, where $b \neq 0$, then its conjugate, $a-bi$ is also a root
  - (d) Every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors
- What does it all mean → we can solve EVERY polynomial (it may be REALLY difficult, but it can be done!)

(F) Using the FTA

- Write an equation of a polynomial whose roots are x = 1, x = 2 and x = ¾
- Write the equation of the polynomial whose roots are 1, -2, -4, & 6 and a point (-1, -84)
- Write the equation of a polynomial whose roots are x = 2 (with a multiplicity of 2) as well as $x = -1 \underline{+} \sqrt{2}$
- Write the equation of a polynomial whose graph is given:



(F) Using the FTA

- Given that $1 - 3i$ is a root of $x^4 - 4x^3 + 13x^2 - 18x - 10 = 0$, find the remaining roots.
- Write an equation of a third degree polynomial whose given roots are 1 and *i*. Additionally, the polynomial passes through (0,5)
- Write the equation of a quartic wherein you know that one root is $2 - i$ and that the root x = 3 has a multiplicity of 2.

(G) Further Examples

- The equation $x^3 - 3x^2 - 10x + 24 = 0$ has roots of 2, *h*, and *k*. Determine a quadratic equation whose roots are $h - k$ and hk.
- The 5th degree polynomial, f(x), is divisible by x^3 and $f(x) - 1$ is divisible by $(x - 1)^3$. Find f(x).

(H) Nth Roots of Unity

- Find the cubed roots of unity
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- ➔ i.e. solve $x = \sqrt[3]{1}$

(H) Nth Roots of Unity

- Find the cubed roots of unity ➔ $x = \sqrt[3]{1}$
- Cube both sides & create a polynomial

$$x = \sqrt[3]{1}$$
$$(x)^3 = \left(\sqrt[3]{1}\right)^3$$
$$x^3 = 1$$
$$x^3 - 1 = 0$$

- So now solve $x^3 - 1 = 0$

$$(x-1)\left(x^2 + x + 1\right) = 0$$

(I) Further Examples - Challenge

- Find the polynomial p(x) with integer coefficients such that one solution of the equation p(x)=0 is 1+√2+√3 .

(I) Further Examples - Challenge

- Start with the linear polynomial: y = −3x + 9. The x-coefficient, the root and the intercept are -3, 3 and 9 respectively, and these are in arithmetic progression. Are there any other linear polynomials that enjoy this property?
- What about quadratic polynomials? That is, if the polynomial $y = ax^2 + bx + c$ has roots r_1 and r_2 can a, r_1, b, r_2 and c be in arithmetic progression?