Lesson 13 – Working with Polynomial Equations

HL1 Math - Santowski

Lesson Objectives

- Mastery of the factoring of polynomials using the algebraic processes of synthetic division
- Mastery of the algebraic processes of solving polynomial equations by factoring (Factor Theorem & Rational Root Theorem)
- Mastery of the algebraic processes of solving polynomial inequalities by factoring (Factor Theorem & Rational Root Theorem)

(A) FAST FIVE

- Factor $x^2 - x - 2$
- Explain what is meant by the term “factor of a polynomial”
- Explain what is meant by the term “root of a polynomial”
- Divide $x^3 - x^2 - 14x + 24$ by $x - 2$
- Divide $x^3 - x^2 - 14x + 24$ by $x + 3$

(A) Roots & Factors

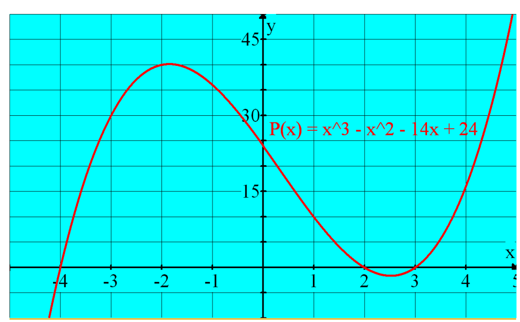
- In our work with quadratics, we saw the “factored” form or “intercept” form of a quadratic equation/expression
- i.e. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$ ➔ factored form of eqn
- So when we solve $f(x) = 0$ ➔ $0 = (x - 2)(x + 1)$, we saw that the zeroes/x-intercepts/roots were $x = 2$ and $x = -1$
- So we established the following connection:
- Factors ➔ $(x - 2)$ and $(x - 1)$
- Roots ➔ $x = 2$ and $x = 1$
- So we will now reiterate the following connections:
- If $(x - R)$ is a ***factor*** of $P(x)$, then $x = R$ is ***root*** of $P(x)$

  AND THE CONVERSE
- If $x = R$ is a ***root*** of $P(x)$, then $(x - R)$ is a ***factor*** of $P(x)$

(B) Review

- Divide $x^3 - x^2 - 14x + 24$ by $x - 2$ and notice the remainder
- Then evaluate $P(2)$. What must be true about $(x - 2)$?
- Divide $x^3 - x^2 - 14x + 24$ by $x + 3$ and notice the remainder
- Then evaluate $P(-3)$. What must be true about $(x+3)$?
- Now graph $f(x) = x^3 - x^2 - 14x + 24$ and see what happens at $x = 2$ and $x = -3$
- So our conclusion is: $x - 2$ is a factor of $x^3 - x^2 - 14x + 24$, whereas $x + 3$ is not a factor of $x^3 - x^2 - 14x + 24$

(B) Review – Graph of $P(x) = x^3 - x^2 - 14x + 24$



(C) The Factor Theorem

- We can use the ideas developed in the review to help us to draw a connection between the polynomial, its factors, and its roots.
- What we have seen in our review are the key ideas of the Factor Theorem - in that if we know a root of an equation, we know a factor and the converse, that if we know a factor, we know a root.
- The Factor Theorem is stated as follows: ***x - a* is a factor of *f(x)* if and only if *f(a) = 0***. To expand upon this idea, we can add that ***ax - b* is a factor of *f(x)* if and only if *f(b/a) = 0***.
- Working with polynomials, $(x + 1)$ is a factor of $x^2 + 2x + 1$ because when you divide $x^2 + 2x + 1$ by $x + 1$ you get a 0 remainder and when you substitute $x = -1$ into $x^2 + 2x + 1$, you get 0

(D) Examples

- ex 1. Show that $x - 2$ is a factor of $x^3 - 7x + 6$
- ex. 2. Show that -2 is a root of $2x^3 + x^2 - 2x + 8 = 0$. Find the other roots of the equation. (Show with GDC)
- ex. 3. Factor $x^3 + 1$ completely
- ex. 4. Factor $x^3 – 1$ completely
- ex. 4. Is $x - \sqrt{2}$ a factor of $x^4 – 5x^2 + 6$?

(D) Further Examples

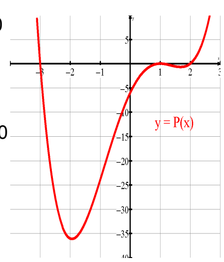
- Ex 1 ➔ Factor $P(x) = 2x^3 – 9x^2 + 7x + 6$ & then solve $P(x) = 0$
- Hence, sketch $y = |P(x)|$
- Solve $2x^3 – 9x^2 + 7x + 6 < 0$
- Ex 2 ➔ Factor $3x^3 – 7x^2 + 8x – 2$ & then solve $P(x) = 0$
- Hence, sketch $y = |P(x)|$
- Solve $3x^3 – 7x^2 + 8x \geq 2$
- Ex 3 ➔ Factor & solve $f(x) = 3x^3 + x^2 – 22x – 24$
- Hence, sketch $y = |P(x)|$
- Solve $-3x^3 \leq x^2 – 22x – 24$

(D) Further Examples - Quartics

- Solve $x^4 - x^3 - 7x^2 + 13x - 6 = 0$
- Hence solve $x^4 - x^3 + 13x \geq 6 + 7x^2$
- Hence, sketch a graph of $P(x) = | x^4 - x^3 - 7x^2 + 13x - 6 |$
- Hence, sketch a graph of the reciprocal of $x^4 - x^3 - 7x^2 + 13x - 6$

(D) Further Examples - Solutions

- Solve $x^4 – x^3 – 7x^2 + 13x – 6 = 0$
- $P(x) = 0 = (x – 1)^2(x + 3)(x – 2)$
- Solve $x^4 – x^3 – 7x^2 + 13x – 6 \geq 0$
- So $P(x) \geq 0$ on $[-\infty,-3)$ or $(2, \infty)$



(E) Factoring Polynomials – The Remainder Theorem

- So in factoring P(x), we use the remainder of the division in order to make a decision about whether or not the x – A is/is not a factor of P(x)
- So if we only want to find a remainder, is there another way (rather than only division?)
- YES there is ➔ it's called the Remainder Theorem

(E) Factoring Polynomials – The Remainder Theorem

- Divide $-2x^3 + 3x^2 - 39x - 20$ by $x + 1$
- Evaluate P(-1). What do you notice?
- What must be true about $x + 1$

- Divide $x^3 – 8x^2 + 11x + 5$ by $x - 2$
- Evaluate P(2). What do you notice?
- What must be true about $(x – 2)$?

- Divide $3x^3 – 4x^2 - 2x - 5$ by $x + 1$
- Evaluate P(-1). What do you notice?

- EXPLAIN WHY THIS WORKS????

(E) Factoring Polynomials – The Remainder Theorem

- Divide $3x^3 – 4x^2 - 2x - 5$ by $x + 1$
- $Q(x) = (3x^2 – 7x + 5)$ with a remainder of -10
- Evaluate P(-1) ➔ equals -10

- EXPLAIN WHY THIS WORKS????

- if rewritten as:
- $3x^3 – 4x^2 - 2x - 5 = (x + 1)(3x^2 - 7x + 5) - 10$,
- $P(-1) = (-1 + 1)(3(-1)5 – 7(-1) + 5) – 10 = -10$

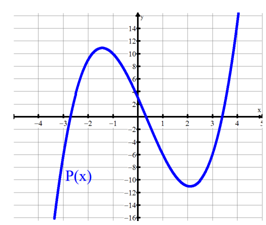
(E) Factoring Polynomials – The Remainder Theorem

- the remainder theorem states "when a polynomial, P(x), is divided by (ax - b), and the remainder contains *no term in x*, then the remainder is equal to P(b/a)

- PROVE WHY THIS IS TRUE ?!?!?!?!?

(F) Factoring Polynomials – The Remainder Theorem - Connection

- So, from the graph on the right, determine the remainder when P(x) is divided by:
- (i) $x + 2$
- (ii) $x – 2$
- (iii) $x – 4$



(G) Factoring Polynomials – The Remainder Theorem - Examples

- Find *k* so that when $x^2 + 8x + k$ is divided by $x - 2$, the remainder is 3

- Find the value of *k* so that when $x^3 + 5x^2 + 6x + 11$ is divided by $x + k$, the remainder is 3

- When $P(x) = ax^3 – x^2 - x + b$ is divided by $x - 1$, the remainder is 6. When P(x) is divided by $x + 2$, the remainder is 9. What are the values of *a* and *b*?

- Use the remainder theorem to determine if $(x – 4)$ is a factor of P(x) if $P(x) = x^4 – 16x^2 – 2x + 6$

(H) Factoring Polynomials – the Rational Root Theorem

- Our previous examples were slightly misleading … as in too easy (leading coefficient was deliberately 1)

- Consider this example ➔ $-12x^3 + 20x^2 + 33x – 20$ which when factored becomes $(2x-1)(3x+4)(5-2x)$ so the roots would be ½, -4/3, and 5/2
- Make the following observation ➔ that the numerator of the roots (1, -4, 5) are factors of the constant term (-20) while the denominator of the roots (2,3,2) are factors of the leading coefficient (-12)

- We can test this idea with other polynomials ➔ we will find the same pattern ➔ that the roots are in fact some combination of the factors of the leading coefficient and the constant term

(H) Factoring Polynomials – the Rational Root Theorem

- Our previous observation (although limited in development) leads to the following theorem:
- Given $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots.. + a_1x^1 + a_0$, if $P(x) = 0$ has a rational root of the form a/b and a/b is in lowest terms, then a must be a divisor of a_0 and b must be a divisor of a_n

(H) Factoring Polynomials – the Rational Root Theorem

- So what does this theorem mean?
- If we want to factor $P(x) = 2x^3 - 5x^2 + 22x - 10$, then we first need to find a value a/b such that $P(a/b) = 0$
- So the factors of the leading coefficient are $\{\pm 1, \pm 2\}$ which are then the possible values for b
- The factors of the constant term, -10, are $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ which are then the possible values for a
- Thus the possible ratios a/b which we can test to find the factors are $\{\pm 1, \pm ½, \pm 2, \pm 5/2, \pm 5, \pm 10\}$
- As it then turns out, $P(½)$ turns out to give $P(x) = 0$, meaning that $(x - ½)$ or $(2x - 1)$ is a factor of $P(x)$

(I) Factoring Polynomials – the Rational Root Theorem - Examples

- Ex 1. To factor $P(x) = 2x^3 - 9x^2 + 7x + 6$, what values of x could you test according to the RRT
- Now factor P(x)
- Ex 2. To factor $P(x) = 3x^3 - 7x^2 + 8x - 2$ what values of x could you test according to the RRT
- Now factor P(x)
- ex 3 → Graph $f(x) = 3x^3 + x^2 - 22x - 24$ using intercepts, points, and end behaviour. Approximate turning points, max/min points, and intervals of increase and decrease.

(J) Conclusion

- So, we now have some "simple" algebra tools that we can use to factor polynomials
- We use the rational root theorem and the remainder theorem
- We use these techniques in order to determine whether a chosen binomial $(ax + b)$ is or is not a factor of our polynomial

(K) Multiplicity of Roots

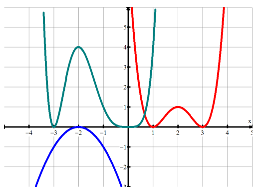
- Factor the following polynomials:
- $P(x) = x^2 - 2x - 15$
- $P(x) = x^2 - 14x + 49$
- $P(x) = x^3 + 3x^2 + 3x + 1$
- Now solve each polynomial equation, $P(x) = 0$
- Solve $0 = 5(x + 1)^2(x - 2)^3$
- Solve $0 = x^4(x - 3)^2(x + 5)$
- Solve $0 = (x + 1)^3(x - 1)^2(x - 5)(x + 4)$

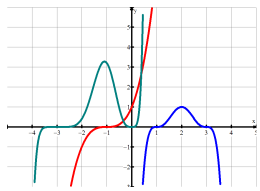
(K) Multiplicity of Roots

- If *r* is a zero of a polynomial and the exponent on the factor that produced the root is *k*, $(x - r)^k$, then we say that *r* has **multiplicity** of *k*. Zeroes with a multiplicity of 1 are often called **simple** zeroes.
- For example, the polynomial $x^2 - 14x + 49$ will have one zero, $x = 7$, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as, $(x - 7)^2 = (x - 7)(x - 7)$
- Written this way the term $(x - 7)$ shows up twice and each term gives the same zero, $x = 7$.
- Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

(K) Multiplicity ➔ Graphic Connection

Even Multiplicity **Odd Multiplicity**



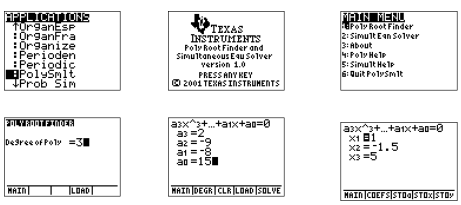


(L) Solving & Factoring on the TI-84

- Use your calculator to answer the following Qs:
- (a) Solve $2x^3 - 9x^2 - 8x = -15$ non-graphically
- (b) Solve $0 = 2x^3 - 9x^2 + 7x + 6$ non-graphically
- (c) Use a graph to factor
  $P(x) = x^4 - x^3 - 7x^2 + 13x - 6$

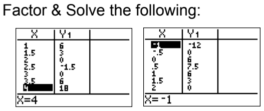
(L) Solving & Factoring on the TI-84

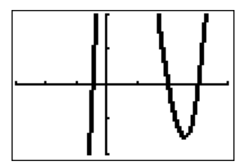
- Solve $2x^3 - 9x^2 - 8x = -15$ turn it into a "root" question ➔ i.e Solve $P(x) = 0$ ➔ Solve $0 = 2x^3 - 9x^2 - 8x + 15$



(L) Solving & Factoring on the TI-84

- Factor & Solve the following:





- $0 = 2x^3 - 9x^2 + 7x + 6$ ➔ roots at x = -0.5,2,3 ➔ would imply factors of $(x - 2)$, $(x - 3)$ and $(x + ½)$ ➔ $P(x) = 2(x + ½)(x - 2)(x - 3)$
- So when factored $P(x) = (2x + 1)(x - 2)(x - 3)$

(L) Factoring Polynomials – The Remainder Theorem - Examples

- Factor the following polynomials using the Remainder Theorem:
- $P(x) = -2x^3 - x^2 + 25x - 12$
- $P(x) = 4x^3 - 12x^2 - 19x + 12$
- $P(x) = 12x^4 + 32x^3 - 15x^2 - 8x + 3$