Lesson 11 – Polynomial Fcns - Working with Quadratic Fcns

HL1 Math - Santowski

Lesson Objectives

- (1) Work with the discriminant and its role
- (2) Sums and Products of Roots
- (3) Quadratic Inequalities
- (4) Introduce the concept of Complex Numbers
  - Find and classify all real and complex roots of a quadratic equation
  - Understand the "need for" an additional number system
  - Add, subtract, multiply, divide, and graph complex numbers
  - Find and graph the conjugate of a complex number

(1) The Discriminant

- the discriminant is the name given to the expression that appears under the square root sign in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

- The discriminant tells you about the "nature" of the roots of a quadratic equation ($ax^2 + bx + c = 0$ given that a,b and c are real numbers
- Three "conditions" for the discriminant ➔

(1) The Discriminant

- the discriminant thus equal to

$$d = \sqrt{b^2 - 4ac}$$

- four "conditions" for the discriminant:
  - $d > 0$ ➔ two real root
  - d is a perfect square ➔real, ratrional roots
  - $d = 0$ ➔ one real root
  - $d < 0$ ➔ no real roots

Examples: Working with the Discriminant

- (1) Find the value(s) for ***M*** such that $2x^2 + Mx + 1 = 0$ has one real solution.
- (2) Find the value(s) for ***K*** such that $x^2 + 4x + K = 0$ has two real solutions.
- (3) Find ***k*** if the equation x2 – (***k*** + 3)x + (***k*** + 6) = 0 has:
  - No real root
  - One real roots
  - Two real roots

Examples: Working with the Discriminant

- Suppose you roll a die with the numbers 1 to 6 on it, and whichever number comes up, substitute that for ***b*** in the equation $x^2 + bx + 4 = 0$
- What is the probability that the resulting equation will have:
  - No solutions?
  - Exactly one solution?
  - Two solutions?
- How would your solutions change if the equation was:
  - $x^2 + bx + 5 = 0$
  - $x^2 + bx + K = 0$
- What if you had 2 die?

(2) Sums and Products of the Roots

- For the quadratic equation $(x - 4)(x + 2) = 0$;
- (a) determine the roots
- (b) determine the:
  - (i) sum of the roots
  - (ii) product of the roots
- (c) expand $(x - 4)(x + 2)$
- (d) connections? (say between b(i), b(ii) and (c) ??)

(2) Sums and Products of the Roots

- Repeat for $(x - 4)(x + 5) = 0$ and $(x - 7)(x + 2) = 0$ and $(2x - 3)(x - 1) = 0$
- (a) determine the roots
- (b) determine the:
  - (i) sum of the roots
  - (ii) product of the roots
- (c) expand $(x - 4)(x + 2)$
- (d) connections? (say between b(i), b(ii) and (c) ??)

(2) Sums and Products of the Roots – Extension of Ideas

- For the cubic equation $(x - 2)(x + 3)(x - 1) = 0$;
- (a) determine the roots
- (b) determine the:
  - (i) sum of the roots
  - (ii) product of the roots
- (c) expand $(x - 2)(x + 3)(x - 1)$ (use wolframalpha.com)
- (d) connections? (b/w b(i), b(ii) and (c) ??)

(2) Sums and Products of the Roots – Extension of Ideas

- For the quartic equation $(x - 2)(x + 3)(x - 1)(x + 1) = 0$;
- (a) determine the roots
- (b) determine the:
  - (i) sum of the roots
  - (ii) product of the roots
- (c) expand $(x - 2)(x + 3)(x - 1)(x + 1)$ (wolframalpha.com)
- (d) connections? (b/w b(i), b(ii) and (c) ??)

Examples: Sums and Products of Roots

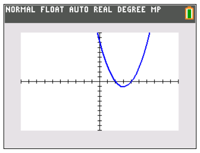
- Without solving the equation, find the sum and products of the roots of $2x^2 - 4x - 12 = 0$
- EXTENDING IDEAS:
- For the equation $x^2 - 2x + 3 = 0$:
  - (i) Find the roots using the algebraic method of your choice
  - (ii) determine the sum of the roots
  - (iii) determine the product of the roots

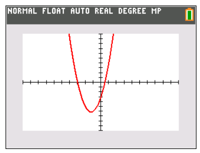
(3) Quadratic Inequalities

- WE will work with three simple inequalities to outline various strategies for working with quadratic inequalities:
- (i) $x^2 - 6x + 8 > 0$
- (ii) $2x^2 + 5x - 3 \leq 0$
- (iii) $3 - x^2 < 2x$
- Strategy #1 ➔ use GRAPH
- Strategy #2 ➔ use vertex & direction of opening
- Strategy #3 ➔ use algebra (arrggghhhhh)

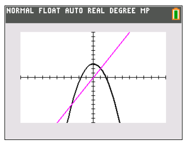
Working with Quadratic Inequalities

- (i) $x^2 - 6x + 8 > 0$ (ii) $2x^2 + 5x - 3 \leq 0$





- (iii) $3 - x^2 < 2x$



Working with Quadratic Inequalities

- (i) $x^2 - 6x + 8 > 0$
  - Zeroes at x = 2 and 4
  - Optimal value is at f(3) = -1
  - Opens up
- (ii) $2x^2 + 5x - 3 \leq 0$
  - zeroes at -3 and 0.5
  - optimal at f(-1.25) = -49/8
  - opens up
- (iii) $3 - x^2 < 2x$ ➔ rearrange as $0 < x^2 + 2x - 3$
  - Zeroes at x = -3, 1
  - Optimal value at f(-1) = -4
  - Opens up

Working with Quadratic Inequalities

- if ab > 0 (and a and b are real numbers)
- Then EITHER: (i) a>0 and b>0 OR (ii) a<0 and b<0
- SO ….
- (i) $x^2 - 6x + 8 > 0$
  - Factors as $(x - 4)(x - 2)$
- (ii) $2x^2 + 5x - 3 \leq 0$
  - factored to $(2x - 1)(x + 3)$
- (iii) $3 - x^2 < 2x$ ➔ rearrange as $0 < x^2 + 2x - 3$
  - Factored to $(x + 3)(x - 1)$
- EXTENSION: what about abc > 0 and $(x - 2)(x + 1)(x - 4) > 0$??

Fast Five

- STORY TIME…..
- http://mathforum.org/johnandbetty/frame.htm

(A) Introduction to Complex Numbers

- Solve the equation $x^2 - 1 = 0$
- We can solve this many ways (factoring, quadratic formula, completing the square & graphically)
- In all methods, we come up with the solution $x = \pm 1$, meaning that the graph of the quadratic has 2 roots at $x = \pm 1$.
- Now solve the equation $x^2 + 1 = 0$

(A) Introduction to Complex Numbers

- Now solve the equation $x^2 + 1 = 0$
- The equation $x^2 = -1$ has no roots because you cannot take the square root of a negative number.
- Long ago mathematicians decided that this was too restrictive.
- They did not like the idea of an equation having no solutions -- so they invented them.
- They proved to be very useful, even in practical subjects like engineering.

(A) Introduction to Complex Numbers

- Consider the quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$.
- The usual formula obtained by ``completing the square'' gives the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are "happy".

(A) Introduction to Complex Numbers

- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are happy.
- If $b^2 < 4ac$ (or if $b^2 - 4ac < 0$) then the number under the square root is negative and you would say that the equation has no solutions.
- In this case we write $b^2 - 4ac = (-1)(4ac - b^2)$ and $4ac - b^2 > 0$. So, in an obvious formal sense,

$$x = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a}$$

- and now the only `meaningless' part of the formula is $\sqrt{-1}$

(A) Introduction to Complex Numbers

- So we might say that any quadratic equation either has ``real'' roots in the usual sense or else has roots of the form $p \pm q\sqrt{-1}$ where p and q belong to the real number system .
- The expressions $p \pm q\sqrt{-1}$ do not make any sense as real numbers, but there is nothing to stop us from playing around with them as **symbols** as $p + qi$ (but we will use $a + bi$)
- We call these numbers complex numbers; the special number i is called an imaginary number, even though i is just as ``real'' as the real numbers and complex numbers are probably simpler in many ways than real numbers.

(B) Using Complex Numbers → Solving Equations

- Note the difference (in terms of the expected solutions) between the following 2 questions:

- Solve $x^2 + 2x + 5 = 0$ where $x \in R$
- Solve $x^2 + 2x + 5 = 0$ where $x \in C$

(B) Using Complex Numbers → Solving Equations

- Solve the following quadratic equations where $x \in C$
- Simplify all solutions as much as possible
- Rewrite the quadratic in factored form

- $x^2 - 2x = -10$
- $3x^2 + 3 = 2x$
- $5x = 3x^2 + 8$
- $x^2 - 4x + 29 = 0$

- What would the "solutions" of these equations look like if $x \in R$

(B) Using Complex Numbers → Solving Equations

- Now that you have solved these equations where $x \in C$

- $x^2 - 2x = -10$
- $3x^2 + 3 = 2x$
- $5x = 3x^2 + 8$
- $x^2 - 4x + 29 = 0$

- Now verify your solutions algebraically!!!

(B) Using Complex Numbers ➔ Solving Equations

- One root of a quadratic equation is $2 + 3i$
- (a) What is the other root?
- (b) What are the factors of the quadratic?
- (c) If the y-intercept of the quadratic is 6, determine the equation in factored form and in standard form.
- (d) Determine the sum and product of the roots