AP Calculus

CHAPTER 4 WORKSHEET

APPLICATIONS OF DIFFERENTIATION

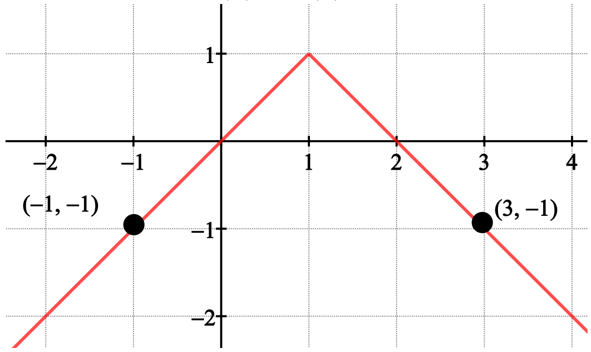
Name________________________________

Seat # ______ Date ____________________

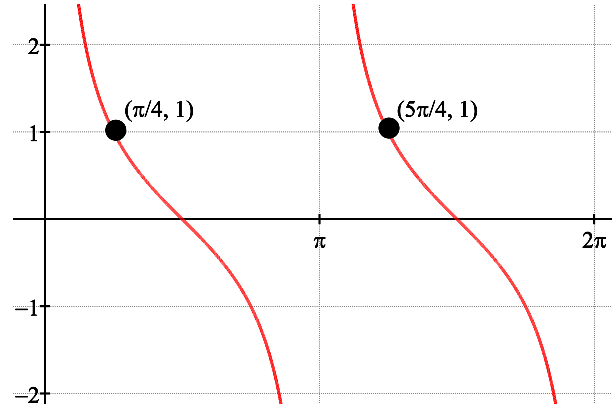
MVT and Rolle's Theorem

UNLESS INDICATED, DO NOT USE YOUR CALCULATOR FOR ANY OF THESE QUESTIONS

In problems 1 and 2, state why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

1. 

$f(x) = 1 - |x - 1|$

2. 

$f(x) = \cot x$

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3. Determine whether the Mean Value Theorem (MVT) applies to the function $f(x) = 3x^2 - x$ on the interval [–1, 2]. If it applies, find all the value(s) of c guaranteed by the MVT for the indicated interval.

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4. Determine whether the MVT applies to the function $f(x) = \dfrac{x+1}{x}$ on the interval [–2, 3]. If it applies, find all the value(s) of c guaranteed by the MVT for the indicated interval.

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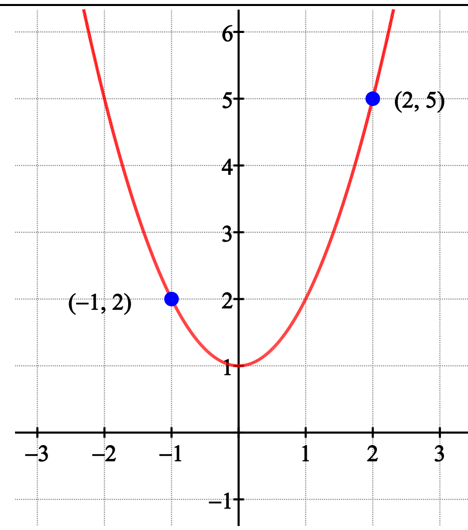
5. Consider the graph of the function $g(x) = x^2 + 1$ shown to the right.
   a) On the drawing provided, draw the secant line through the points $(-1, 2)$ and $(2, 5)$.
   b) Since g is both continuous and differentiable, the MVT guarantees the existence of a tangent line(s) to the graph parallel to the secant line. Sketch such line(s) on the drawing.
   c) Use your sketch from part (b) to visually estimate the x-coordinate at the point of tangency.
   d) That x-coordinate at the point of tangency is the value of c promised by the MVT. Verify your answer to part (c) by using the conclusion of the MVT on the interval $[-1, 2]$ to find c.



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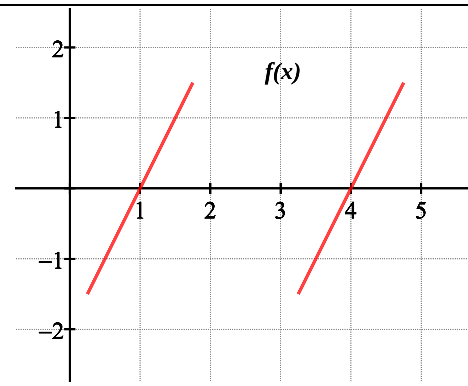
6. Given $h(x) = x^{2/3}$, explain why the hypothesis of the MVT are met on $[0, 8]$ but are not met on $[-1, 8]$.

The MVT and Rolle's Theorem are not the only existence theorems we have seen this year. The Intermediate Value Theorem (IVT) is also an important existence theorem for Calculus (see below.)

Intermediate Value Theorem (IVT):	
Conditions:	let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let k be any number between $f(a)$ and $f(b)$.
Consequence:	there is at least one value c in $[a, b]$ such that $f(c)=k$.

Use the appropriate theorem (either IVT or MVT or Rolle's) to answer the remaining problems.

7. The graph to the right shows portions of a continuous function f.
   a) Explain why f must have a root on the interval $(1, 4)$.
   b) <u>Must</u> f have a horizontal tangent line on the interval $(1, 4)$? Explain why or why not.
   c) What additional condition should the function f meet to guarantee the existence of a horizontal tangent line on the interval $(1, 4)$?



For questions 8 through 12, determine whether the statements <u>must</u> be true, <u>might</u> be true, or <u>cannot</u> be true. Justify your answers.

8. If $f(1)<0$ and $f(5)>0$, then there must be a number c in $(1, 5)$ such that $f(c)=0$.

9. If f is continuous on $[1, 5]$ and $f(1)<0$ and $f(5)>0$, then there must be a number c in $(1, 5)$ such that $f(c)=0$.

10. If f is continuous on $[1, 5]$ and $f(1)=2$ and $f(5)=2$, then there must be a number c in $(1, 5)$ such that $f'(c)=0$.

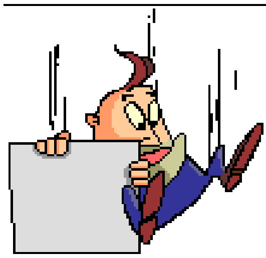
11. If f is differentiable on $[1, 5]$ and $f(1)=2$ and $f(5)=2$, then there must be a number c in $(1, 5)$ such that $f'(c)=0$.

12. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.

Some of the following questions require using the IVT and MVT "backwards." This means that a fact is stated and you need to identify what theorem was used to guarantee that fact. You might want to read again the conclusions for the IVT and MVT before attempting these problems!

13. Given $h(x)=x^3+x-1$ on the interval $[0, 2]$, will there be a value p such that $0<p<2$ and $h'(p)=5$? Justify your answer. If your answer is yes, find p.

14. Given $g(x)=x^3-x^2+x$ on the interval [1, 3], will there be a value r such that $1<r<3$ and $g(r)=11$? Justify your answer. If your answer is yes, use your calculator to find r.



You may use a calculator for this problem.

15. The height of an object t seconds after it is dropped from a height of 500 meters is $h(t)=-4.9t^2+500$.

a) Find the average velocity of the object during the first three seconds. Remember: average velocity is equal to change in position divided by change in time.

b) Show that at some time during the first three seconds of fall the instantaneous velocity must equal the average velocity you found in part (a). Then, find that time.

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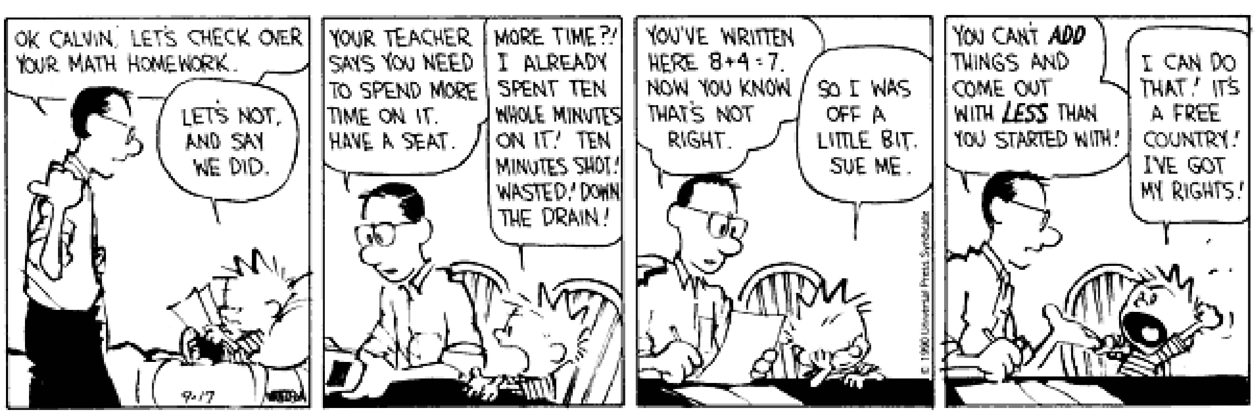
16. The functions f and g are twice differentiable for all real numbers. The table below shows values of the functions and their derivatives at selected values of x. The function h is given by $h(x)=f(x)g(x)$.

x	***f(x)***	***g(x)***	***f'(x)***	***g'(x)***
1	6	4	−4	3
2	9	−1	−1	1
3	5	1	0	−2
4	−3	3	5	6

a) Explain why there must be a value r for $1<r<4$ such that $h(r)=-2$.

b) Explain why there must be a value p for $1<p<4$ such that $h'(p)=-11$.

c) Is the function h increasing or decreasing when $x=3$? Justify your answer.



AP Calculus

CHAPTER 4 WORKSHEET

APPLICATIONS OF DIFFERENTIATION

ANSWER KEY

MVT and Rolle's Theorem

1. Rolle's Theorem does not apply because $f(x)=1-|x-1|$ is not differentiable for all values on the interval (–1, 3). Its derivative does not exist at $x = 1$.

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2. Rolle's Theorem does not apply because $f(x)=\cot x$ is not continuous for all values on the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. It is discontinuous at $x=\pi$.

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3. Since $f(x)=3x^2-x$ is a polynomial, it is both continuous and differentiable on the interval $[-1, 2] \Rightarrow$ MVT applies.
$$\frac{f(2)-f(-1)}{2-(-1)}=\frac{10-4}{3}=6c-1 \Rightarrow c=\frac{1}{2}$$

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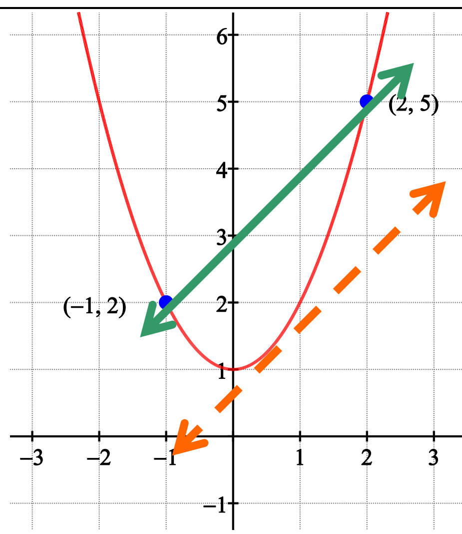
4. Since $f(x)=\frac{x+1}{x}$ is discontinuous at $x = 0$, the MVT does not apply on the interval $[-2, 3]$.

5.
   a) See drawing (on solid green.)
   b) See drawing (on dashed orange.)
   c) $x \approx \frac{1}{2}$
   d) $\frac{g(2)-g(-1)}{2-(-1)}=\frac{5-2}{3}=2c \Rightarrow c=\frac{1}{2}$



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6. Since $h'(x)=\frac{2}{3}\cdot x^{-1/3}=\frac{2}{3x^{1/3}}$, $h'(x)$ does not exist (that is, *h(x)* is not differentiable) at $x = 0$. Therefore, *h(x)* is continuous in $[0, 8]$ and differentiable in $(0, 8) \Rightarrow$ MVT applies. However, *h(x)* is continuous in $[-1, 8]$ but not differentiable in $(-1, 8) \Rightarrow$ MVT does not apply.

7.
   a) Since *f(x)* is continuous, the IVT applies. Since there are values where $f(x)>0$ and values where $f(x)<0$, there must be a value c where $f(c)=0$.
   b) No, not necessarily. We could just connect the graph with a line segment making *f(x)* continuous but not differentiable.
   c) *f(x)* should be differentiable.

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8. Might be true, but if the function is not continuous it could be false.

9. The IVT guarantees this is always true.

10. Might be true, but if the function is not differentiable it could be false.

11. The MVT or Rolle's Theorem guarantee this is always true.

12. Might be true, but if the function is not differentiable it could be false.

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13. Since $h(x) = x^3 + x - 1$ is a polynomial, it is both continuous and differentiable on the interval $[0, 2] \Rightarrow$ MVT applies. Using it: $\dfrac{h(2)-h(0)}{2-(0)} = \dfrac{9-(-1)}{2} = 5 = h'(p)$. So, yes such value exists.

$5 = 3p^2 + 1 \Rightarrow p = \pm\dfrac{2}{\sqrt{3}}$. However, since p must be inside $(0, 2) \Rightarrow p = \dfrac{2}{\sqrt{3}}$

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14. Since $g(x) = x^3 - x^2 + x$ is a polynomial, it is continuous on the interval $[1, 3] \Rightarrow$ IVT applies. Since: $g(3) = 21$ and $g(1) = 1 \Rightarrow g(x)$ will reach all values between 1 and 21, including 11. So, yes such value exists.
$11 = r^3 - r^2 + r \Rightarrow r \approx 2.439$

15.
a) Average velocity $= \dfrac{h(3)-h(0)}{3-0} = -14.7$ ft/sec

b) Since $h(t) = -4.9t^2 + 500$ is a polynomial, it is both continuous and differentiable on the interval $[0, 3] \Rightarrow$ MVT applies. Using it: $\dfrac{h(3)-h(0)}{3-(0)} = h'(c)$. So the average velocity, $\dfrac{h(3)-h(0)}{3-(0)}$, will equal the instantaneous velocity, $h'(t)$, at some time c.

$= \dfrac{h(3)-h(0)}{3-0} = -14.7 = -9.8c \Rightarrow c = 1.5$ sec

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16.
a) Since f and g are continuous functions (because they are differentiable), so is $h(x) = f(x)\cdot g(x) \Rightarrow$ Intermediate Value Theorem applies to $h(x)$ in $[1, 4]$. With

$$\left.\begin{aligned} h(1) &= f(1)\cdot g(1) = 6\cdot 4 = 24 \\ h(4) &= f(4)\cdot g(4) = (-3)\cdot 3 = -9 \end{aligned}\right\} \Rightarrow \text{there must be a value } r \text{ for}$$

$1 < r < 4$ such that $-9 = h(4) < h(r) = -2 < h(1) = 24$.

b) Since f and g are continuous and differentiable functions, so is $h(x) = f(x)\cdot g(x) \Rightarrow$ Mean Value Theorem applies to $h(x)$ in $[1, 4]$. With

$$\left.\begin{aligned} h(1) &= f(1)\cdot g(1) = 6\cdot 4 = 24 \\ h(4) &= f(4)\cdot g(4) = (-3)\cdot 3 = -9 \end{aligned}\right\} \Rightarrow \frac{h(4)-h(1)}{4-1} = -11$$

So there must be a value p for $1 < p < 4$ such that $h'(p) = \dfrac{h(4)-h(1)}{4-1} = -11$.

c) $h'(x) = f'(x)g(x) + f(x)g'(x) \Rightarrow h'(3) = -10 \Rightarrow h$ is decreasing at $x = 3$ because $h'(3) < 0$.
