8.2 Applying the Sine Law

YOU WILL NEED

- ruler

GOAL

Use the sine law to calculate unknown side lengths and angle measures in acute triangles.

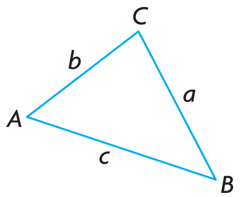
LEARN ABOUT the Math

In Lesson 8.1, you discovered the **sine law** for acute triangles. Can you be sure that the sine law is true for every acute triangle?

sine law

in any acute triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

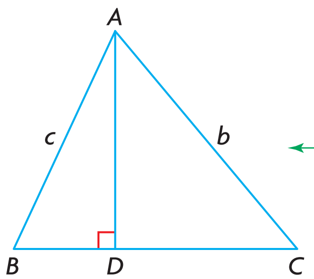


? How can you show that the ratio $\frac{\text{length of opposite side}}{\sin\text{(angle)}}$ is the same for all three angle-side pairs in any acute triangle?

EXAMPLE 1 **Proving** the sine law for acute triangles

Show that the sine law is true for all acute triangles.

Ben's Solution



I drew an acute triangle. Since the primary trigonometric ratios are used for right triangles only, I drew a perpendicular height, *AD*, from *A* to *BC* to form two right triangles, △*ABD* and △*ACD*.

In △*ABD*,

$$\sin B = \frac{AD}{c}$$

In △*ACD*,

$$\sin C = \frac{AD}{b}$$

Since the sine law involves the sine of the angles in △*ABC*, I wrote equations for the sines of ∠*B* and ∠*C* in the two right triangles using the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$.

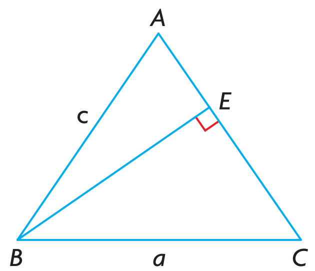
$$c \sin B = AD \qquad b \sin C = AD$$

I knew that *AD* was the opposite side in both right triangles, so I used the sine ratio to describe *AD* in △*ABD* and △*ACD*. I thought this would allow me to relate the two triangles.

$$c \sin B = b \sin C$$
$$\frac{c \sin B}{\sin C} = b$$
$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

Since the expressions $c \sin B$ and $b \sin C$ both describe *AD*, I set them equal.

I wanted to write each side of the equation as a ratio with information about only one triangle. I divided both sides of the equation by $\sin C$ and then by $\sin B$.



In $\triangle ABE$,
$$\sin A = \frac{BE}{c}$$
$$c \sin A = BE$$

In $\triangle CBE$,
$$\sin C = \frac{BE}{a}$$
$$a \sin C = BE$$

I wanted to show that the ratios $\frac{c}{\sin C}$ and $\frac{b}{\sin B}$ were equal to $\frac{a}{\sin A}$.

To do this, I needed to divide $\triangle ABC$ differently so that I could use $\angle A$. I reasoned that if I drew a height *BE* from *B* to *AC*, I could follow the same steps I used for height *AD*.

$$c \sin A = a \sin C$$
$$c = \frac{a \sin C}{\sin A}$$
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

Since the expressions $c \sin A$ and $a \sin C$ both describe *BE*, I set them equal. Then I divided both sides of the equation by $\sin A$ and then by $\sin C$.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Since $\frac{b}{\sin B}$ and $\frac{a}{\sin A}$ are equal to $\frac{c}{\sin C}$, all three ratios must be equal.

Reflecting

A. Why did Ben need to draw line segment *AD* perpendicular to side *BC*?

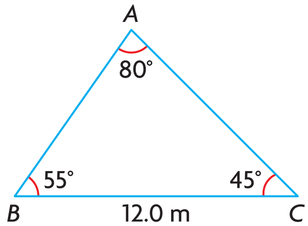
B. If Ben drew a perpendicular line segment from vertex *C* to side *AB*, which pair of ratios in the sine law do you think he could show are equal?

C. Why does it make sense that the sine law can also be written in the form $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$?

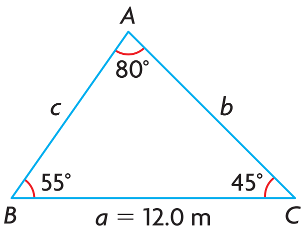
APPLY the Math

EXAMPLE 2 **Selecting a sine law strategy** to calculate the length of a side

Determine the length of *AC*.



Elizabeth's Solution



I named the sides of the triangle using lower-case letters that correspond to the opposite angles. Then I identified the measures I didn't know.

Since the triangle does not contain a right angle, I couldn't use the primary trigonometric ratios.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{12.0}{\sin 80°} = \frac{b}{\sin 55°}$$

$$\sin 55°\left(\frac{12.0}{\sin 80°}\right) = \sin 55°\left(\frac{b}{\sin 55°}\right)$$

$$\sin 55°\left(\frac{12.0}{\sin 80°}\right) = b$$

$$0.8192\left(\frac{12.0}{0.9848}\right) \doteq b$$

$$9.98 \doteq b$$

I realized that I could use the sine law if I knew an opposite side-angle pair, plus one more side or angle in the triangle. I knew a and $\angle A$ and I wanted to know b, so I related a, b, $\sin A$, and $\sin B$. I could have used $\frac{\sin A}{a} = \frac{\sin B}{b}$, but I decided to use $\frac{a}{\sin A} = \frac{b}{\sin B}$ since I had to solve for b. This meant that b was the numerator and I could multiply both sides by sin 55° to solve for b.

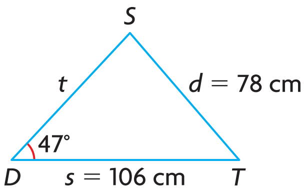
The length of *AC* is about 10.0 m.

It made sense that b is shorter than a, since $\angle B$ is less than $\angle A$.

EXAMPLE 3 **Selecting a sine law strategy** to calculate the measure of an angle

In $\triangle DST$, $\angle D = 47°$, $d = 78$ cm, and $s = 106$ cm. Determine the measure of $\angle S$.

Phil's Solution



I drew a diagram. I knew that $\angle S$ was greater than $\angle D$ since $s > d$. I couldn't assume that the triangle contained a right angle, so I couldn't use the primary trigonometric ratios.

$$\frac{\sin S}{s} = \frac{\sin D}{d}$$

$$\frac{\sin S}{106} = \frac{\sin 47°}{78}$$

$$106\left(\frac{\sin S}{106}\right) = 106\left(\frac{\sin 47°}{78}\right)$$

$$\sin S = 106\left(\frac{\sin 47°}{78}\right)$$

$$\sin S \doteq 0.9939$$

I knew s, d, and $\angle D$, and I wanted to determine the measure of $\angle S$. So, I used the sine law that included these four quantities. I used the proportion with sin S and sin D in the numerators to make the equation easier to solve for sin S.

$$\angle S = \sin^{-1}(0.9939)$$

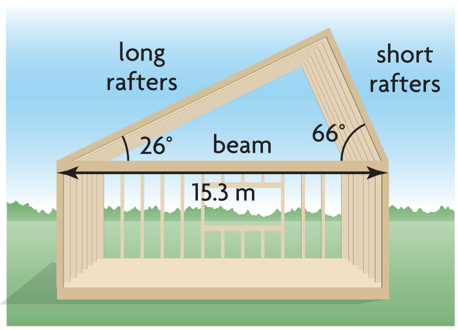
$$\angle S \doteq 83.7°$$

To determine $\angle S$, I calculated the inverse of the sine ratio.

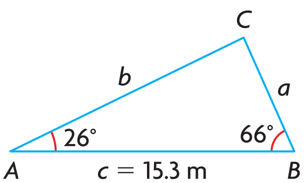
$\angle S$ is about 84°.

EXAMPLE 4 **Solving a problem** using the sine law

The roof of a new house must be built to exact specifications so that solar panels can be installed. The long rafters at the front of the house must be inclined at an angle of 26° to the horizontal beam. The short rafters at the back of the house must be inclined at an angle of 66°. The house is 15.3 m wide. Determine the length of the long rafters.



Taylor's Solution



I drew an acute triangle to model the situation. The long rafters were opposite $\angle B$, the 66° angle, so I labelled them side b. I knew that my diagram was correct, and the long rafters were side b, since $66° > 26°$.

$$\angle C = 180° - 26° - 66°$$
$$= 88°$$

Since I knew length c, I had to use ratios that involved $\angle B$ and $\angle C$. So, I had to determine the measure of $\angle C$. I knew that the angles in a triangle add up to 180°.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 66°} = \frac{15.3}{\sin 88°}$$

Since I needed to determine side b, I used the sine law to write a proportion with the sides in the numerator.

$$\sin 66°\left(\frac{b}{\sin 66°}\right) = \sin 66°\left(\frac{15.3}{\sin 88°}\right)$$

I solved for b.

$$b = \sin 66°\left(\frac{15.3}{\sin 88°}\right)$$
$$b \doteq 13.99$$

Since $\angle B$ is less than $\angle C$, it makes sense that b is shorter than c.

The long rafters are about 14.0 m long.

In Summary

Key Idea

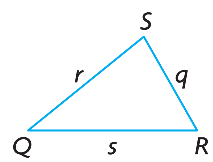
- The sine law can be used to determine unknown side lengths or angle measures in some acute triangles.

Need to Know

- To use the sine law to determine a side length or angle measure, follow these steps:
  - Determine the ratio of the sine of a known angle measure and a known side length.
  - Create an equivalent ratio using the unknown side length and the measure of its opposite angle, or the sine of the unknown angle measure and the length of its opposite side.
  - Equate the ratios you created, and solve.
- You can use the sine law to solve a problem modelled by an acute triangle if you know the measurements of
  - two sides and the angle that is opposite one of these sides
  - two angles and any side
- An acute triangle can be divided into right triangles. The proof of the sine law involves writing proportions that compare corresponding sides in these right triangles.

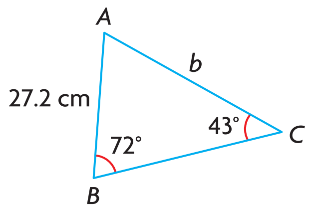
CHECK Your Understanding

1. Write three equivalent ratios using the sides and angles in the triangle at the right.



2. Determine the indicated measures to one decimal place.

a)



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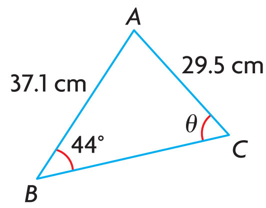
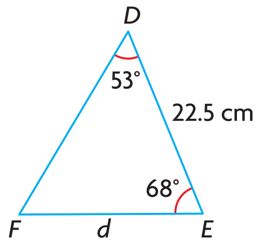


PRACTISING

3. Determine the indicated side lengths and angle measures.

a)



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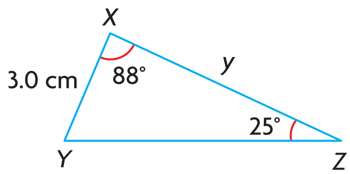

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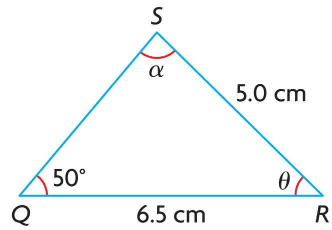

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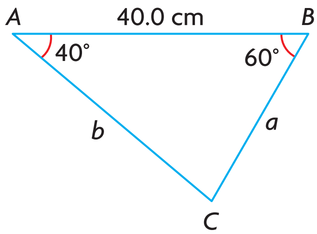

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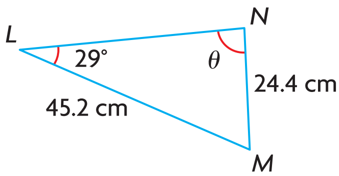

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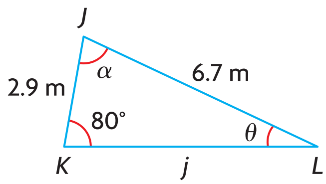
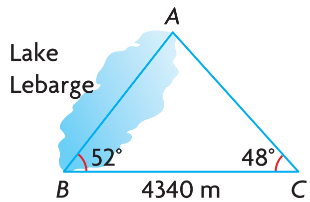


4. Scott is a naturalist. He is studying the
A effects of acid rain on fish populations in different lakes. As part of his research, he needs to know the length of Lake Lebarge. Scott makes the measurements shown. How long is Lake Lebarge?



5. Draw a labelled diagram for each triangle. Then calculate the required side length or angle measure.

a) In $\triangle SUN$, $n = 58$ cm, $\angle N = 38°$, and $\angle U = 72°$. Determine the length of side u.

b) In $\triangle PQR$, $\angle R = 73°$, $\angle Q = 32°$, and $r = 23$ cm. Determine the length of side q.

c) In $\triangle TAM$, $t = 8$ cm, $m = 6$ cm, and $\angle T = 65°$. Determine the measure of $\angle M$.

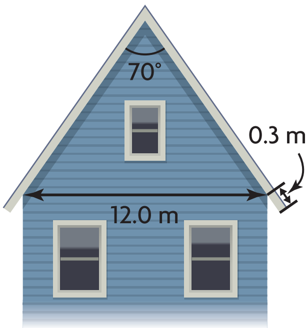
d) In $\triangle WXY$, $w = 12.0$ cm, $y = 10.5$ cm, and $\angle W = 60°$. Determine the measure of $\angle Y$.

6. In $\triangle CAT$, $\angle C = 32°$, $\angle T = 81°$, and $c = 24.1$ m.
K Solve the triangle.

Environment *Connection*

Acidic lakes cannot support the variety of life in healthy lakes. Clams and crayfish are the first to disappear, followed by other species of fish.

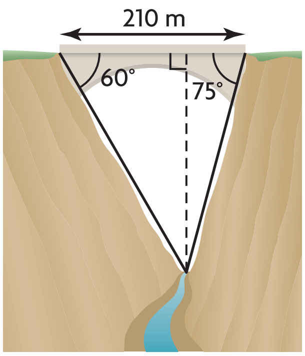
7. The short sides of a parallelogram are both 12.0 cm. The acute angles of the parallelogram are 65°, and the short diagonal is 15.0 cm. Determine the length of the long sides of the parallelogram. Round your answer to the nearest tenth of a centimetre.



8. An architect designed a house that is 12.0 m wide. The rafters that hold up the roof are equal in length and meet at an angle of 70°, as shown at the left. The rafters extend 0.3 m beyond the supporting wall. How long are the rafters?

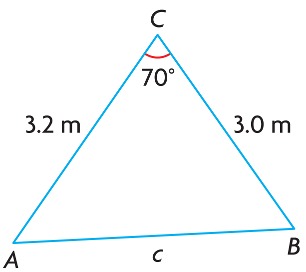
9. A telephone pole is supported by two wires on opposite sides. **T** At the top of the pole, the wires form an angle of 60°. On the ground, the ends of the wires are 15.0 m apart. One wire makes a 45° angle with the ground. How long are the wires, and how tall is the pole?

10. In $\triangle PQR$, $\angle Q = 90°$, $r = 6$, and $p = 8$. Explain two different ways to calculate the measure of $\angle P$.



11. A bridge across a gorge is 210 m long, as shown in the diagram at the left. The walls of the gorge make angles of 60° and 75° with the bridge. Determine the depth of the gorge to the nearest metre.

12. Use the sine law to help you describe each situation.
   C a) Three pieces of information allow you to solve for all the unknown side lengths and angle measures in a triangle.
   b) Three pieces of information do not allow you to solve a triangle.



13. Jim says that the sine law cannot be used to determine the length of side c in $\triangle ABC$ at the left. Do you agree or disagree? Explain.

14. Suppose that you know the length of side p in $\triangle PQR$, as well as the measures of $\angle P$ and $\angle Q$. What other sides and angles could you calculate? Explain how you would determine these measurements.

Extending

15. In $\triangle ABC$, $\angle A = 58°$, $\angle C = 74°$, and $b = 6$. Calculate the area of $\triangle ABC$ to one decimal place.

16. An isosceles triangle has two sides that are 10 cm long and two angles that measure 50°. A line segment bisects one of the 50° angles and ends at the opposite side. Determine the length of the line segment.

17. Use the sine law to write a ratio that is equivalent to each expression for $\triangle ABC$.
   a) $\dfrac{a}{\sin A}$ b) $\dfrac{\sin A}{\sin B}$ c) $\dfrac{a}{c}$ d) $\dfrac{a \sin C}{c \sin A}$