8.2 Compound Interest: Future Value

GOAL

Determine the future value of a principal being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software (optional)

LEARN ABOUT the Math

Mena invests $2000 in a bank account that pays 6%/a compounded annually. The savings account is called the "Accumulator" and pays compound interest.

? What type of function will model the growth of Mena's money?

A. Calculate the interest earned and amount at the end of the first year. Record your answers in a table as shown.

Year	Balance at Start of Year	Interest Earned	Balance at End of Year
0	—	—	$2000
1	$2000		
2			
3			
4			
5			

B. Complete the table for the 2nd to 5th years.

C. Enter your data for Year and Balance at End of Year into either lists on a graphing calculator or columns in a spreadsheet.

D. Create a scatter plot, using Year as the independent variable.

E. What type of function best models the growth of Mena's money? You may need to calculate more data points before you decide. Explain how you know.

F. Determine the function that models the amount of her investment over time.

compound interest
interest that is added to the principal *before* new interest earned is calculated. So interest is calculated on the principal *and* on interest already earned. Interest is paid at regular time intervals called the **compounding period.**

compounding period
the intervals at which interest is calculated; for example,
annually ⇒ 1 time per year
semi-annually ⇒ 2 times per year
quarterly ⇒ 4 times per year
monthly ⇒ 12 times per year

Reflecting

G. Compare the total amount of Mena's investment with that based on the same principal earning simple interest. What is the advantage of earning compound interest over simple interest?

H. How are compound interest, exponential functions, and geometric sequences related?

APPLY the Math

EXAMPLE 1 Representing any situation earning compound interest as a function

Tim borrows \$5300 at 4.6%/a compounded annually.

a) How much will he have to pay back if he borrows the money for 10 years?

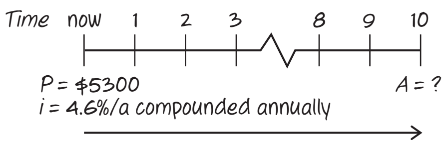
b) Determine the future value, A, and interest earned, I, if he invested a principal of \$$P$ for n years at i%/a compounded annually.

future value

the total amount, A, of an investment after a certain length of time

Shelley's Solution

a)



I drew a timeline for the situation.

end of 1st year:

$$A = P(1 + rt)$$
$$= 5300[1 + 0.046(1)]$$
$$= 5300(1.046)$$
$$= \$5543.80$$

At the end of the first year, Tim gets charged 4.6% of his original \$5300 loan. So I calculated the amount he would owe at the end of that year.

end of 2nd year:

$$A = 5543.80[1 + 0.046(1)]$$
$$= 5543.80(1.046)$$
$$\doteq \$5798.81$$

At the end of the second year, he gets charged 4.6% of the amount he owed at the end of the first year, \$5543.80. So I calculated the amount he would owe at the end of that year.

end of 3rd year:

$$A = 5798.81[1 + 0.046(1)]$$
$$= 5798.81(1.046)$$
$$= \$6065.56$$

I used the same method to calculate the amount at the end of the third year. Each time, I rounded to the nearest cent.

$$t_1 = 5300 \times 1.046^1 = \$5543.80$$
$$t_2 = 5300 \times 1.046^2 \doteq \$5798.81$$
$$t_3 = 5300 \times 1.046^3 \doteq \$6065.56$$
$$\vdots$$
$$t_n = 5300 \times 1.046^n$$

I noticed that I was multiplying by 1.046 each time. This is a geometric sequence with common ratio 1.046. The general term of the sequence is $t_n = 5300 \times 1.046^n$. I used this formula to calculate the first three terms, and I got the same numbers as in my previous calculations.

$$t_{10} = 5300 \times 1.046^{10}$$
$$\doteq \$8309.84$$

To determine how much Tim would owe after 10 years, I substituted $n = 10$ into the formula for the general term.

Tim would have to pay back \$8309.84 after 10 years.

b) end of 1st year:

For compound interest, the amount or future value depends on time, as it did for simple interest. Since the interest rate is *i*%/a, I substituted *i* for *r* into the formula for the total amount and calculated the future value for the 1st, 2nd, and 3rd years.

$$A = P(1 + in)$$

$$A_1 = P[1 + i(1)]$$

$$= P(1 + i)$$

end of 2nd year:

$$A_2 = [P(1 + i)](1 + i)$$

$$= P(1 + i)^2$$

end of 3rd year:

$$A_3 = [P(1 + i)^2](1 + i)$$

$$= P(1 + i)^3$$

end of *n*th year:

This is a geometric sequence with first term $P(1 + i)$ and common ratio $1 + i$, so I wrote the general term, which gave me the amount after *n* years. The amount is an exponential function in terms of time.

$$A = P(1 + i)^n$$

To determine the interest earned over a period of *n* years, I subtracted the principal from the total amount.

$$I = A - P$$

$$= P(1 + i)^n - P$$

$$= P[(1 + i)^n - 1]$$

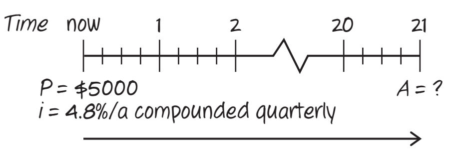
I factored out the common factor *P*.

The future value of an investment of $*P* for *n* years at *i*%/a compounded annually will be $A = P(1 + i)^n$, and the total interest earned will be $I = P[(1 + i)^n - 1]$.

EXAMPLE 2 Selecting a strategy to determine the amount when the compounding period is less than a year

Lara's grandparents invested $5000 at 4.8%/a compounded quarterly when she was born. How much will the investment be worth on her 21st birthday?

Herman's Solution



I drew a timeline for the situation.

$P = \$5000$

$i = 0.048 \div 4$

$= 0.012$

Since interest is paid quarterly for each compounding period, I divided the annual interest rate by 4 to get the interest rate.

$n = 21 \times 4$

$= 84$

Interest is paid 4 times per year, so I calculated the number of compounding periods. This is the total number of times that the interest would be calculated over the 21 years.

$A = P(1 + i)^n$

I used the formula $A = P(1 + i)^n$, where i is the interest rate per compounding period and n is the number of compounding periods.

$A = 5000(1 + 0.012)^{84}$

$\doteq \$13\,618.62$

I noticed that solving this problem is the same as solving a problem in which the money earns 1.2%/a compounded annually for 84 years.

The $5000 investment will be worth $13 618.62 on Lara's 21st birthday.

EXAMPLE 3 Calculating the difference of the amounts of two different investments

On her 15th birthday, Trudy invests $10 000 at 8%/a compounded monthly. When Lina turns 45, she invests $10 000 at 8%/a compounded monthly. If both women leave their investments until they are 65, how much more will Trudy's investment be worth?

Henry's Solution

$P = \$10\,000$

$i = \frac{0.08}{12}$

To calculate how much Trudy's investment will be worth when she turns 65, I first determined the interest rate per month as a fraction.

$n = (65 - 15) \times 12$

$= 600$

Since interest is compounded monthly and she is investing for 50 years, there will be $50 \times 12 = 600$ compounding periods.

$A = 10\,000\left(1 + \frac{0.08}{12}\right)^{600}$

$\doteq \$538\,781.94$

Trudy's investment will be worth $538 781.94 when she turns 65.

$P = \$10\ 000$

$i = \dfrac{0.08}{12}$

Lina's investment has the same principal and interest rate per month as Trudy's, but fewer compounding periods.

$n = (65 - 45) \times 12$

$= 240$

Since Lina invested for 20 years, there will be $20 \times 12 = 240$ compounding periods.

$A = 10\ 000\left(1 + \dfrac{0.08}{12}\right)^{240}$

$\doteq \$49\ 268.03$

Lina's investment will be worth \$49 268.03 when she turns 65.

$\$538\ 781.94 - \$49\ 268.03 = \$489\ 513.91$

I subtracted Lina's amount from Trudy's amount.

Trudy's investment will be worth \$489 513.91 more than Lina's.

EXAMPLE 4 Comparing simple interest and compound interest

Nicolas invests \$1000. How long would it take for his investment to double for each type of interest earned?

a) 5%/a simple interest

b) 5%/a compounded semi-annually

Jesse's Solution

a) $P = \$1000$

$r = 5\% = 0.05$

I knew the principal and the interest rate.

$I = \$1000$

$I = Prt$

Since Nicolas's investment will double and he is earning simple interest, the interest earned must be the same as the principal.

$1000 = 1000(0.05)t$

I substituted the values of P, r, and I into the formula for the interest earned.

$\dfrac{1000}{1000(0.05)} = \left(\dfrac{1000(0.05)}{1000(0.05)}\right)t$

To solve for t, I divided both sides of the equation by 1000(0.05).

$20 = t$

It will take 20 years for Nicolas's investment to double at 5%/a simple interest.

b) $$i = \frac{0.05}{2} = 0.025$$

$$A = P(1 + i)^n$$

Since interest is paid semi-annually, I divided the annual interest rate by 2 to get the interest rate per half year.

$$A = 1000(1 + 0.025)^{40}$$
$$\doteq \$2685.06$$

I substituted $P = 1000$ and $i = 0.025$ into the formula for the amount. Then I used guess-and-check to determine n. I tried 20 years, or $n = 20 \times 2 = 40$ compounding periods.

$$A = 1000(1 + 0.025)^{20}$$
$$\doteq \$1638.62$$

The amount after 20 years was too much.

Next I tried 10 years, or $n = 10 \times 2 = 20$ compounding periods, but that wasn't enough.

$$A = 1000(1 + 0.025)^{28}$$
$$\doteq \$1996.50$$

Since my second guess was slightly closer to \$2000 than my first guess, I tried 14 years, or $n = 14 \times 2 = 28$ compounding periods. The result was close to double.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	"=B2 + 1000*0.05/2"	"=C2*(1 + 0.025)"
4	1	"=B3 + 1000*0.05/2"	"=C3*(1 + 0.025)"

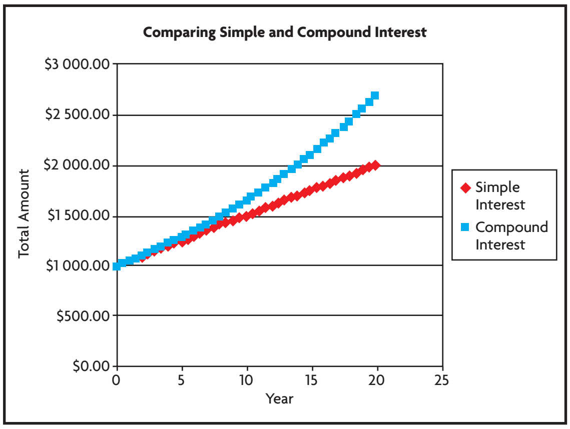
I then used a spreadsheet to check my result.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	$1 025.00	$1 025.00
4	1	$1 050.00	$1 050.63
5	1.5	$1 075.00	$1 076.89
6	2	$1 100.00	$1 103.81
7	2.5	$1 125.00	$1 131.41
8	3	$1 150.00	$1 159.69

28	13	$1 650.00	$1 900.29
29	13.5	$1 675.00	$1 947.80
30	14	$1 700.00	$1 996.50
31	14.5	$1 725.00	$2 046.41

40	19	$1 950.00	$2 555.68
41	19.5	$1 975.00	$2 619.57
42	20	$2 000.00	$2 685.06

The investment earning simple interest took 20 years to double. If Nicolas earns compound interest, he gets interest on interest previously earned, so his investment grows faster. I used the spreadsheet to compare the two possibilities. With compound interest, the investment almost doubles after 14 years.



It will take about 14 years for Nicolas's investment to double at 5%/a compounded semi-annually.

I graphed the amount of the investment for both cases. From the graph, the simple-interest situation is modelled by a linear function growing at a constant rate, while the compound-interest situation is modelled by an exponential function growing at a changing rate.

Tech *Support*

For help using a spreadsheet to graph functions, see Technical Appendix, B-21.

In Summary

Key Ideas

- Compound interest is calculated by applying the interest rate to the principal and any interest already earned.
- The total amounts at the end of each interest period form a geometric sequence. So compound interest results in exponential growth.
- The total amount, A, of an investment after a certain period is called the future value of the investment.

Need to Know

- Banks pay or charge compound interest at regular intervals called the compounding period. If interest is compounded annually, then at the end of the first year, interest is calculated and added to the principal. At the end of the second year, interest is calculated on the new balance (principal plus interest earned from the previous year). This pattern continues every year the investment is kept.
- The future value of an investment earning compound interest can be calculated using the formula $A = P(1 + i)^n$, where A is the future value; P is the principal; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.
- The most common compounding periods are:

annually	1 time per year	i = annual interest rate	n = number of years
semi-annually	2 times per year	i = annual interest rate ÷ 2	n = number of years × 2
quarterly	4 times per year	i = annual interest rate ÷ 4	n = number of years × 4
monthly	12 times per year	i = annual interest rate ÷ 12	n = number of years × 12

- Compound interest can be calculated using the formula $I = A - P$ or $I = P[(1 + i)^n - 1]$, where I is the total interest.

CHECK Your Understanding

1. Copy and complete the table.

	Rate of Compound Interest per Year	Compounding Period	Time	Interest Rate per Compounding Period, *i*	Number of Compounding Periods, *n*
a)	5.4%	semi-annually	5 years		
b)	3.6%	monthly	3 years		
c)	2.9%	quarterly	7 years		
d)	2.6%	weekly	10 months		

2. **i)** Determine the amount owed at the end of each of the first five compounding periods.

ii) Determine the general term for the amount owed at the end of the *n*th compounding period.

	Amount Borrowed	Rate of Compound Interest per Year	Compounding Period
a)	$10 000	7.2%	annually
b)	$10 000	3.8%	semi-annually
c)	$10 000	6.8%	quarterly
d)	$10 000	10.8%	monthly

3. Calculate the future value of each investment. Draw a timeline for each.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$258	3.5%	annually	10 years
b)	$5 000	6.4%	semi-annually	20 years
c)	$1 200	2.8%	quarterly	6 years
d)	$45 000	6%	monthly	25 years

PRACTISING

4. For each investment, determine the future value and the total interest earned.
K

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$4 000	3%	annually	4 years
b)	$7 500	6%	monthly	6 years
c)	$15 000	2.4%	quarterly	5 years
d)	$28 200	5.5%	semi-annually	10 years
e)	$850	3.65%	daily	1 year
f)	$2 225	5.2%	weekly	47 weeks

5. Sima invests some money in an account that earns a fixed rate of interest compounded annually. The amounts of the investment at the end of the first three years are shown at the right.

a) Determine the annual rate of compound interest earned.

b) How much did Sima invest?

Year	Total Amount
1	\$4240.00
2	\$4494.40
3	\$4764.06

6. Chris invests \$10 000 at 7.2%/a compounded monthly. How long will it take for his investment to grow to \$25 000?

7. **A** Serena wants to borrow \$15 000 and pay it back in 10 years. Interest rates are high, so the bank makes her two offers:

- Option 1: Borrow the money at 10%/a compounded quarterly for the full term.
- Option 2: Borrow the money at 12%/a compounded quarterly for 5 years and then renegotiate the loan based on the new balance for the last 5 years.

If, in 5 years, the interest rate will be 6%/a compounded quarterly, how much will Serena save by choosing the second option?

8. Ted used the exponential function $A(n) = 5000 \times 1.0075^{12n}$ to represent the future value, A, in dollars, of an investment. Determine the principal, the annual interest rate, and the compounding period. Explain your reasoning.

9. Margaret can finance the purchase of a \$949.99 refrigerator one of two ways:

- Plan A: 10%/a simple interest for 2 years
- Plan B: 5%/a compounded quarterly for 2 years

Which plan should she choose? Justify your answer.

10. Eric bought a \$1000 Canada Savings Bond that earns 5%/a compounded annually. Eric can redeem the bond in 7 years. Determine the future value of the bond.

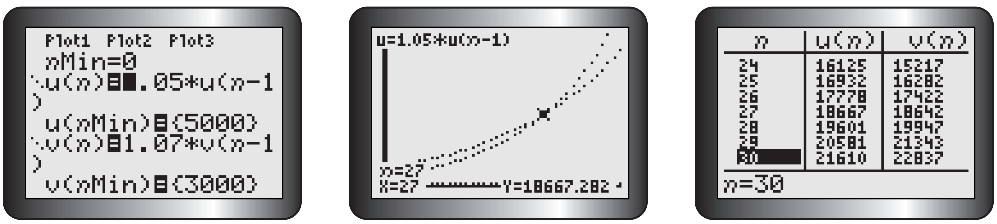
11. Dieter deposits \$9000 into an account that pays 10%/a compounded quarterly. After three years, the interest rate changes to 9%/a compounded semi-annually. Calculate the value of his investment two years after this change.

12. Cliff has some money he wants to invest for his retirement. He is offered two options:

- 10%/a simple interest
- 5%/a compounded annually

Under what conditions should he choose the first option?

13. **T** Noreen used her graphing calculator to investigate two sequences. Three screenshots from her investigation are shown. Create a problem for this situation and solve it.



14. You are searching different banks for the best interest rate on an investment, and you find these rates:
- 6.6%/a compounded annually
- 6.55%/a compounded semi-annually
- 6.5%/a compounded quarterly
- 6.45/a compounded monthly

Rank the rates from most to least return on your investment.

15. On July 1, 1996, Anna invested $2000 in an account that earned 6%/a compounded monthly. On July 1, 2001, she moved the total amount to a new account that paid 8%/a compounded quarterly. Determine the balance in her account on January 1, 2008.

16. Bernie deposited $4000 into an account that pays 4%/a compounded quarterly during the first year. The interest rate on this account is then increased by 0.2% each year. Calculate the balance in Bernie's account after three years.

17. On the day Rachel was born, her grandparents deposited $500 into a savings account that earns 4.8%/a compounded monthly. They deposited the same amount on her 5th, 10th, and 15th birthdays. Determine the balance in the account on Rachel's 18th birthday.

18. **C** Create a mind map for the concept of *interest*. Show how the calculations of simple and compound interest are related to functions and sequences.

Extending

19. Liz decides to save money to buy an electric car. She invests $500 every 6 months at 6.8%/a compounded semi-annually. What total amount of money will she have at the end of the 10th year?

20. An effective annual interest rate is the interest rate that is equivalent to the given one, assuming that compounding occurs annually. Calculate the effective annual interest rate for each loan. Round to two decimal places.

	Rate of Compound Interest per Year	Compounding Period
a)	6.3%	semi-annually
b)	4.2%	monthly
c)	3.2%	quarterly