3.1 Properties of Quadratic Functions

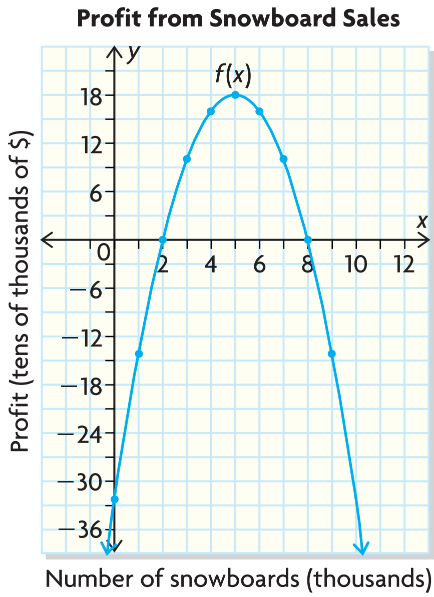
GOAL

Represent and interpret quadratic functions in a number of different forms.

LEARN ABOUT the Math

Francisco owns a business that sells snowboards. His accountants have presented him with data on the business' profit in a table and a graph.

Snowboards Sold, *x* (1000s)	0	1	2	3	4	5	6	7	8	9
Profit, *f*(*x*) ($10 000s)	−32	−14	0	10	16	18	16	10	0	−14



? What function models Francisco's profit?

EXAMPLE 1 Selecting a strategy to describe the algebraic model

Develop an algebraic expression for the function that models Francisco's profit from selling snowboards.

Kelly's Solution: Using the Vertex Form of the Quadratic Function

Snowboards Sold (1000s)	Profit ($10 000s)	First Differences	Second Differences
0	−32		
		18	
1	−14		−4
		14	
2	0		−4
		10	
3	10		−4
		6	
4	16		−4
		2	
5	18		−4
		−2	
6	16		−4
		−6	
7	10		−4
		−10	
8	0		−4
		−14	
9	−14		

This function looks quadratic, since its graph appears to be a parabola. To make sure, I checked the first and second differences. Since the first differences are not constant, the function is nonlinear. The second differences are all equal and negative. So the function is quadratic. This confirms that the graph is a parabola that opens downward.

From the graph, the vertex is (5, 18).
The parabola also passes through (2, 0).

I could determine the quadratic function model if I knew the vertex and at least one other point on the graph.

$$f(x) = a(x - h)^2 + k$$

$$= a(x - 5)^2 + 18$$

I used the **vertex form** of the quadratic function and substituted the coordinates of the vertex from the graph.

$$0 = a(2 - 5)^2 + 18$$

$$0 = 9a + 18$$

$$-18 = 9a$$

$$-2 = a$$

$f(2) = 0$, so I substituted (2, 0) into the function. Once I did that, I solved for a.

The function $f(x) = -2(x - 5)^2 + 18$ models Francisco's profit.

Jack's Solution: Using the Factored Form of the Quadratic Function

The graph is a parabola, opening down with axis of symmetry $x = 5$.

The graph looks like a parabola, so it has to be a quadratic function. The graph is symmetric about the line $x = 5$.

The x-intercepts are the points $(2, 0)$ and $(8, 0)$.

I could find the **factored form** of the quadratic function if I knew the x-intercepts, or zeros. I read these from the graph and used the table of values to check.

I took the factored form of a quadratic function and substituted the values of the x-intercepts for r and s.

$$f(x) = a(x - r)(x - s)$$
$$f(x) = a(x - 2)(x - 8)$$
$$10 = a(3 - 2)(3 - 8)$$
$$10 = a(1)(-5)$$
$$10 = -5a$$
$$-2 = a$$

I then chose the point (3, 10) from the table of values and substituted its coordinates into $f(x)$ to help me find the value of a.

The function $f(x) = -2(x - 2)(x - 8)$ models Francisco's profit.

Reflecting

A. What information do you need to write the vertex form of the quadratic function?

B. What information do you need to write the factored form of the quadratic function?

C. Use the graph to state the domain and range of the function that models Francisco's profit. Explain.

D. Will both of the models for Francisco's profit lead to the same function when expressed in **standard form**?

APPLY the Math

EXAMPLE 2 Determining the properties of a quadratic function

A construction worker repairing a window tosses a tool to his partner across the street. The height of the tool above the ground is modelled by the quadratic function $h(t) = -5t^2 + 20t + 25$, where $h(t)$ is height in metres and t is the time in seconds after the toss.

a) How high above the ground is the window?

b) If his partner misses the tool, when will it hit the ground?

c) If the path of the tool's height were graphed, where would the axis of symmetry be?

d) Determine the domain and range of the function in this situation.

André's Solution

a) $h(0) = -5(0)^2 + 20(0) + 25$

$= 25$

The height of the window must be the value of the function at $t = 0$ s.

The window is 25 m above the ground.

b) $0 = -5t^2 + 20t + 25$

$0 = -5(t^2 - 4t - 5)$

$0 = -5(t + 1)(t - 5)$

If the partner missed the tool, it would hit the ground. The height at the ground is zero. I set $h(t)$ equal to zero. Then I factored the quadratic.

$t = -1$ or $t = 5$

I found two values for t, but the negative answer is not possible, since time must be positive.

The tool will hit the ground after 5 s.

c) $t = \frac{-1 + 5}{2}$

$t = \frac{4}{2}$

$t = 2$

The axis of symmetry passes through the midpoint of the two zeros of the function. I added the zeros together and divided by two to find that t-value. The axis of symmetry is a vertical line, so its equation is $t = 2$.

The axis of symmetry is $t = 2$.

d) Domain $= \{t \in \mathbf{R} \mid 0 \le t \le 5\}$

In this situation, t must be 0 or greater, and the tool will stop when it hits the ground.

$h(2) = -5(2)^2 + 20(2) + 25$

$= -20 + 40 + 25$

$= 45$

Range $= \{h \in \mathbf{R} \mid 0 \le h \le 45\}$

The least possible value of $h(t)$ is zero. I found the value of h at $t = 2$, since the greatest value is the y-coordinate of the vertex, which is always on the axis of symmetry.

EXAMPLE 3 Graphing a quadratic function from the vertex form

Given $f(x) = 2(x - 1)^2 - 5$, state the vertex, axis of symmetry, direction of opening, y-intercept, domain, and range. Graph the function.

Sacha's Solution

Vertex: $(1, -5)$

The x-coordinate of the vertex is 1 and the y-coordinate is -5.

Axis of symmetry: $x = 1$

The axis of symmetry is a vertical line through the vertex at $(1, -5)$.

Direction of opening: up

Since a is positive, the parabola opens up.

$$\begin{aligned} f(0) &= 2(0 - 1)^2 - 5 \\ &= 2(-1)^2 - 5 \\ &= 2 - 5 \\ &= -3 \end{aligned}$$

I substituted $x = 0$ to calculate the y-intercept and solved the equation.

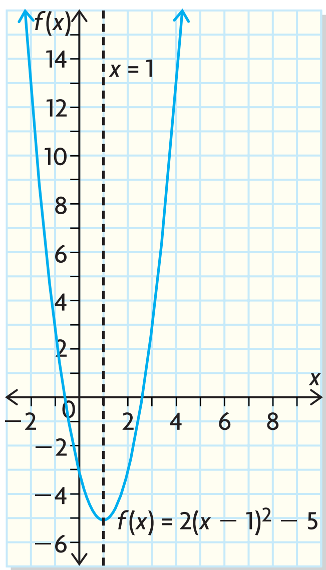
y-intercept: -3

Domain: $\{x \in \mathbf{R}\}$

There are no restrictions on the values for x.

Range: $\{y \in \mathbf{R} \mid y \geq -5\}$

Because the vertex has a y-value of -5 and the parabola opens up, the y-values have to be greater than or equal to -5.



To graph the function, I plotted the vertex and the axis of symmetry.

I found the values of $f(x)$ when $x = 2$, 3, and 4.

$f(2) = -3$

$f(3) = 3$

$f(4) = 13$

The values had to be the same for 0, -1, and -2 because the graph is symmetric about the line $x = 1$.
I plotted the points and joined them with a smooth curve.

In Summary

Key Ideas

- Graphs of quadratic functions with no domain restrictions are parabolas.
- Quadratic functions have constant nonzero second differences. If the second differences are positive, the parabola opens up and the coefficient of x^2 is positive. If the second differences are negative, the parabola opens down and the coefficient of x^2 is negative.

Need to Know

- Quadratic functions can be represented by equations in function notation, by tables of values, or by graphs.
- Quadratic functions have a degree of 2.
- Quadratic functions can be expressed in different algebraic forms:
  - standard form: $f(x) = ax^2 + bx + c, a \neq 0$
  - factored form: $f(x) = a(x - r)(x - s), a \neq 0$
  - vertex form: $f(x) = a(x - h)^2 + k, a \neq 0$

CHECK Your understanding

1. Determine whether each function is linear or quadratic. Give a reason for your answer.

a)

x	*y*
−2	15
−1	11
0	7
1	3
2	−1

b)

x	*y*
−2	1
−1	3
0	6
1	10
2	15

c)

x	*y*
−2	4
−1	8
0	12
1	16
2	20

d)

x	*y*
−2	7
−1	4
0	3
1	4
2	7

2. State whether each parabola opens up or down.

a) $f(x) = 3x^2$

b) $f(x) = -2(x - 3)(x + 1)$

c) $f(x) = -(x + 5)^2 - 1$

d) $f(x) = \frac{2}{3}x^2 - 2x - 1$

3. Given $f(x) = -3(x - 2)(x + 6)$, state

a) the zeros

b) the direction of opening

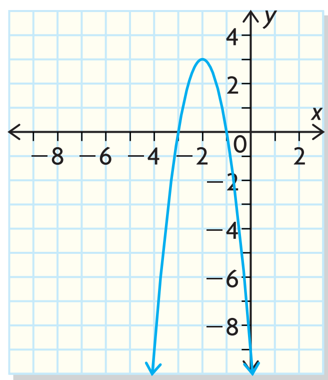
c) the equation of the axis of symmetry

4. Given the parabola at the right, state

a) the vertex

b) the equation of the axis of symmetry

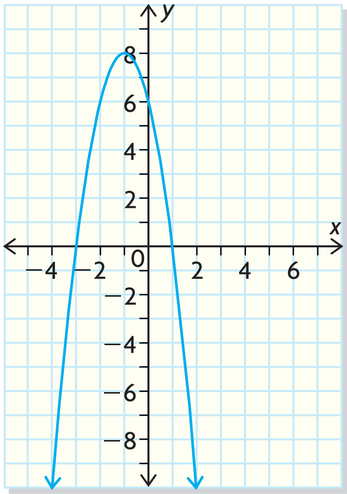
c) the domain and range



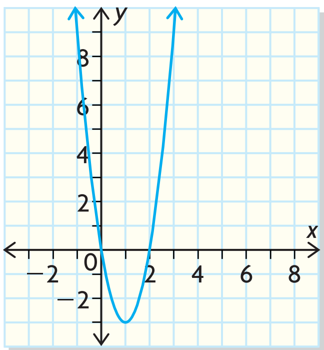
PRACTISING

5. Graph each function. State the direction of opening, the vertex, and the equation of the axis of symmetry.
   a) $f(x) = x^2 - 3$
   b) $f(x) = -(x + 3)^2 - 4$
   c) $f(x) = 2(x - 4)(x + 2)$
   d) $f(x) = -\frac{1}{2}x^2 + 4$

6. Express each quadratic function in standard form. State the *y*-intercept of each.
   a) $f(x) = -3(x - 1)^2 + 6$
   b) $f(x) = 4(x - 3)(x + 7)$

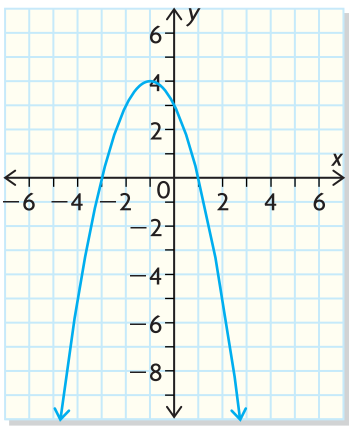


7. Examine the parabola at the left.
   K a) State the direction of opening.
   b) Name the coordinates of the vertex.
   c) List the values of the *x*-intercepts.
   d) State the domain and range of the function.
   e) If you calculated the second differences, what would their sign be? How do you know?
   f) Determine the algebraic model for this quadratic function.



8. Examine the parabola at the left.
   a) State the direction of opening.
   b) Find the coordinates of the vertex.
   c) What is the equation of the axis of symmetry?
   d) State the domain and range of the function.
   e) If you calculated the second differences, what would their sign be? Explain.

9. Each pair of points (x, y) are the same distance from the vertex of their parabola. Determine the equation of the axis of symmetry of each parabola.
   a) $(-2, 2), (2, 2)$
   b) $(-9, 1), (-5, 1)$
   c) $(6, 3), (18, 3)$
   d) $(-5, 7), (1, 7)$
   e) $(-6, -1), (3, -1)$
   f) $\left(-\frac{11}{8}, 0\right), \left(\frac{3}{4}, 0\right)$



10. Examine the parabola shown at the left.
    a) Copy and complete this table.

x	−2	−1	0	1	2
f(*x*)					

    b) Calculate the second differences of the function. How could you have predicted their signs?
    c) Determine the equation of the function.

11. The height of a rocket above the ground is modelled by the quadratic function $h(t) = -4t^2 + 32t$, where $h(t)$ is the height in metres t seconds after the rocket was launched.

A

a) Graph the quadratic function.

b) How long will the rocket be in the air? How do you know?

c) How high will the rocket be after 3 s?

d) What is the maximum height that the rocket will reach?

12. A quadratic function has these characteristics:

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- $x = -1$ is the equation of the axis of symmetry.
- $x = 3$ is the x-intercept.
- $y = 32$ is the maximum value.

Determine the y-intercept of this parabola.

13. Describe two ways in which the functions $f(x) = 2x^2 - 4x$ and $g(x) = -(x - 1)^2 + 2$ are alike, and two ways in which they are different.

C

Extending

14. The first differences and second differences of a quadratic function with domain ranging from $x = -2$ to $x = 3$ are given. If $f(-2) = 19$, copy the table and complete the second row by determining the missing values of the function.

x	−2	−1	0	1	2	3
***f*(*x*)**	19					
First Differences		−10	−6	−2	2	6
Second Differences		4	4	4	4	

15. A company's profit, in thousands of dollars, on sales of computers is modelled by the function $P(x) = -2(x - 3)^2 + 50$, where x is in thousands of computers sold. The company's profit, in thousands of dollars, on sales of stereo systems is modelled by the function $P(x) = -(x - 2)(x - 7)$, where x is in thousands of stereo systems sold. Calculate the maximum profit the business can earn.

16. Jim has a difficult golf shot to make. His ball is 100 m from the hole. He wants the ball to land 5 m in front of the hole, so it can roll to the hole. A 20 m tree is between his ball and the hole, 40 m from the hole and 60 m from Jim's ball. With the base of the tree as the origin, write an algebraic expression to model the height of the ball if it just clears the top of the tree.