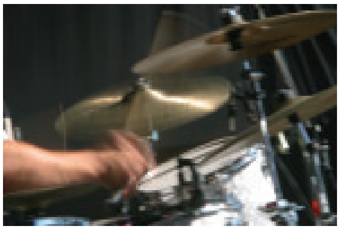
6.6 Solving Problems Using Quadratic Models

YOU WILL NEED

- grid paper
- ruler
- graphing calculator



GOAL

Solve problems that can be modelled by quadratic relations using a variety of tools and strategies.

LEARN ABOUT the Math

The volunteers at a food bank are arranging a concert to raise money. They have to pay a set fee to the musicians, plus an additional fee to the concert hall for each person attending the concert. The relation $P = -n^2 + 580n - 48\,000$ models the profit, P, in dollars, for the concert, where n is the number of tickets sold.

? How can you determine the number of tickets that must be sold to break even and to maximize the profit?

EXAMPLE 1 **Selecting a strategy** to analyze a quadratic relation

Calculate the number of tickets they must sell to break even. Determine the number of tickets they must sell to maximize the profit.

Jack's Solution: Completing the square

$P = -n^2 + 580n - 48\,000$
$P = -(n^2 - 580n) - 48\,000$
$P = -(n^2 - 580n + 84\,100 - 84\,100) - 48\,000$
$P = -[(n - 290)^2 - 84\,100] - 48\,000$
$P = -(n - 290)^2 + 84\,100 - 48\,000$
$P = -(n - 290)^2 + 36\,100$

I completed the square to write the relation in vertex form so I could determine the maximum profit first.

The volunteers must sell 290 tickets to earn a maximum profit of $36 100 for the food bank.

Since $a < 0$, the parabola opens downward. The y-coordinate of the vertex (290, 36 100) is the maximum value.

$0 = -(n - 290)^2 + 36\,100$

I set $P = 0$ to calculate the break-even points.

$(n - 290)^2 = 36\,100$
$\sqrt{(n - 290)^2} = \pm\sqrt{36\,100}$
$n - 290 = \pm 190$
$n = 290 + 190$ or $n = 290 - 190$
$n = 480$ $\quad$ $n = 100$

I used inverse operations to solve for n.

The volunteers break even if they sell 480 tickets or 100 tickets.

Dineke's Solution: Factoring the relation

$P = -n^2 + 580n - 48\ 000$
$P = -(n^2 - 580n + 48\ 000)$
$P = -(n - 480)(n - 100)$

I factored the relation to determine the break-even points first.

$0 = -(n - 480)(n - 100)$
$0 = n - 480$ or $0 = n - 100$
$n = 480$ $n = 100$

I knew that the break-even points would occur when the profit equalled zero, so I set $P = 0$. Then I used inverse operations to solve for n.

The volunteers break even if they sell 480 tickets or 100 tickets.

$$n = \frac{480 + 100}{2}$$
$n = 290$

Since the zeros are the same distance from the axis of symmetry, I used them to determine the equation of the axis of symmetry.

$P = -(n - 480)(n - 100)$
$P = -(290 - 480)(290 - 100)$
$P = -(-190)(190)$
$P = 36\ 100$

The equation of the axis of symmetry, $n = 290$, gave the n-coordinate of the vertex. I substituted $n = 290$ into the equation to determine the P-coordinate of the vertex.

Therefore, the volunteers will earn a maximum profit of $36 100 for the food bank if they sell 290 tickets.

Reflecting

A. How are Jack's solution and Dineke's solution the same? How are they different?

B. Will both strategies always work? Why or why not?

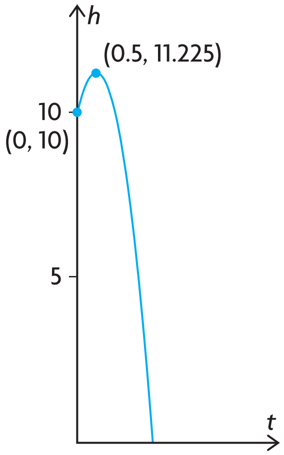
C. Whose strategy would you have used for this problem? Explain your choice.

APPLY the Math

EXAMPLE 2 Solving a problem by creating a quadratic model

Alexandre was practising his 10 m platform dive. Because of gravity, the relation between his height, h, in metres, and the time, t, in seconds, after he dives is quadratic. If Alexandre reached a maximum height of 11.225 m after 0.5 s, how long was he above the water after he dove?

Burns's Solution



I decided to start with a sketch that included the given information. The *y*-intercept is the starting position. The maximum height and time are the coordinates of the vertex, (time, maximum height).

$$y = a(x - h)^2 + k$$
$$10 = a(0 - 0.5)^2 + 11.225$$
$$10 = a(-0.5)^2 + 11.225$$
$$10 - 11.225 = 0.25a$$
$$\frac{-1.225}{0.25} = \frac{0.25a}{0.25}$$
$$-4.9 = a$$
$$h = -4.9(t - 0.5)^2 + 11.225$$

Since I knew the coordinates of the vertex, I determined a model for the height of the diver in vertex form. Substituting the vertex given for (*h*, *k*) and the initial height of the diver for a point (*x*, *y*), I used inverse operations to solve for *a*.

$$0 = -4.9(t - 0.5)^2 + 11.225$$

Since I was determining when Alexandre hit the water, I set $h = 0$.

$$\frac{-11.225}{-4.9} = \frac{-4.9(t - 0.5)^2}{-4.9}$$
$$2.291 \doteq (t - 0.5)^2$$
$$\pm\sqrt{2.291} = \sqrt{(t - 0.5)^2}$$
$$\pm 1.514 \doteq t - 0.5$$
$$0.5 \pm 1.514 \doteq t$$
$$0.5 + 1.514 \doteq t \quad \text{or} \quad 0.5 - 1.514 \doteq t$$
$$2.01 \doteq t \qquad\qquad -1.01 \doteq t$$

I used inverse operations to solve for *t*.

Alexandre was above the water for about 2.0 s after he dove.

The answer -1.01 didn't make sense since time cannot be negative in this situation, so I didn't use it.

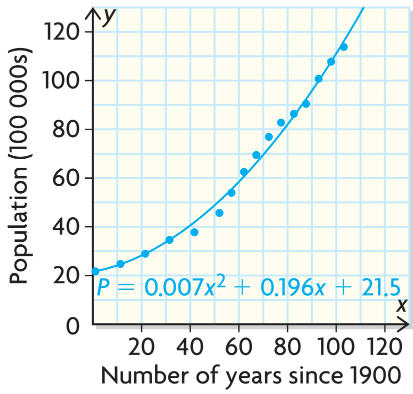
EXAMPLE 3 Reasoning to determine an appropriate solution

Statisticians use various models to make predictions about population growth. Ontario's population (in 100 000s) can be modelled by the relation $P = 0.007x^2 + 0.196x + 21.5$, where x is the number of years since 1900.

a) Using this model, what was Ontario's population in 1925?

b) When will Ontario's population reach 15 million?

Ontario's Population since 1900



Blair's Solution

a) $x = 1925 - 1900$

$x = 25$

$P = 0.007x^2 + 0.196x + 21.5$

$P = 0.007(25)^2 + 0.196(25) + 21.5$

$P = 30.775$

Using x as the number of years since 1900, I subtracted 1900 from 1925. I substituted my result into the relation to determine the population in 1925.

The population was about 3 077 500 in 1925.

The relation gave the population in 100 000s, so I multiplied my answer by 100 000.

b) $P = 0.007x^2 + 0.196x + 21.5$

$150 = 0.007x^2 + 0.196x + 21.5$

The population was given in 100 000s, and 15 million = 150 × 100 000. So, I used 150 for P.

$150 - 150 = 0.007x^2 + 0.196x + 21.5 - 150$

$0 = 0.007x^2 + 0.196x - 128.5$

I rearranged my equation so that I could use the quadratic formula to determine its roots.

$a = 0.007,\ b = 0.196,\ c = -128.5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-0.196 \pm \sqrt{0.196^2 - 4(0.007)(-128.5)}}{2(0.007)}$$

$$x = \frac{-0.196 \pm \sqrt{3.636}}{0.014}$$

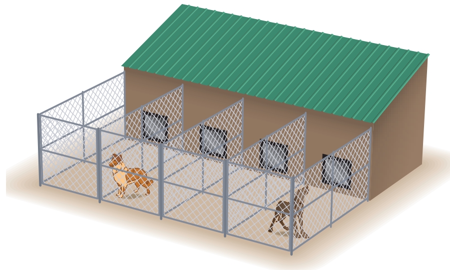
$x \doteq -150.20$ or $x \doteq 122.20$

Ontario's population will be about 15 000 000 in 2022.

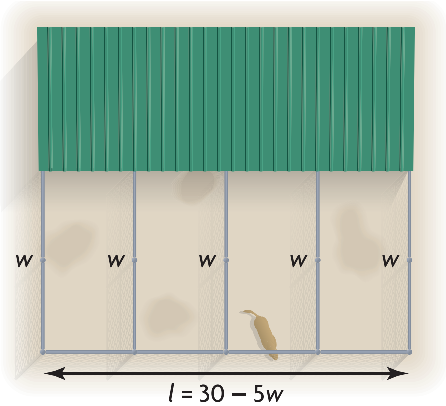
I thought $x = -150.21$ made no sense, since this would mean that the population was 15 000 000 in 1750, which I know is not reasonable. So, I used the other answer and added it to 1900.

EXAMPLE 4 Solving a problem by creating a quadratic model

Lila is creating dog runs for her dog kennel. She can afford 30 m of chain-link fence to surround four dog runs. The runs will be attached to a wall, as shown in the diagram. To achieve the maximum area, what dimensions should Lila use for each run and for the combined enclosure?



Mitsu's Solution



I started by sketching the dog runs. I let w represent the width of each run. I let l represent the total length of the enclosure.

To express the length in terms of the width, I subtracted the amount of fencing needed for the 5 fence sides that are perpendicular to the wall from the amount of fencing Lila can afford.

$A = l \times w$
$A = (30 - 5w) \times w$
$A = 30w - 5w^2$
$A = -5w^2 + 30w$

I wrote an equation for the area and simplified it.

$A = -5(w^2 - 6w)$
$A = -5(w^2 - 6w + 9 - 9)$
$A = -5[(w - 3)^2 - 9]$
$A = -5(w - 3)^2 - (-5)9$
$A = -5(w - 3)^2 + 45$

Since I wanted to maximize the area, I completed the square to determine the vertex.

(3, 45) is the vertex, so the maximum area is 45 m^2. It occurs when the width of each run is 3 m.

$l = 30 - 5w$
$= 30 - 5(3)$
$= 15$

I substituted the width into my expression for the length to determine the length of the combined enclosure.

The dimensions of the combined enclosure should be 15 m by 3 m, and the dimensions of each run should be 3 m by 3.75 m.

$15 \div 4 = 3.75$

In Summary

Key Idea

- When solving a problem that involves a quadratic relation, follow these suggestions:
  - Write the relation in the form that is most helpful for the given situation.
  - Use the vertex form to determine the maximum or minimum value of the relation.
  - Use the standard form or factored form to determine the value of x that corresponds to a given y-value of the relation. You may need to use the quadratic formula.

Need to Know

- A problem may have only one realistic solution, even when the quadratic equation that is used to represent the problem has two real solutions. When you solve a quadratic equation, check to make sure that your solutions make sense in the context of the situation.

CHECK Your Understanding

1. For each relation, explain what each coordinate of the vertex represents and what the zeros represent.

a) a relation that models the height, h, of a ball that has been kicked from the ground after time t

b) a relation that models the height, h, of a ball when it is a distance of x metres from where it was thrown from a second-floor balcony

c) a relation that models the profit earned, P, on an item at a given selling price, s

d) a relation that models the cost, C, to create n items using a piece of machinery

e) a relation that models the height, h, of a swing above the ground during one swing, t seconds after the swing begins to move forward

For questions 2 to 17, round all answers to two decimal places, where necessary.

2. A model rocket is shot straight up from the roof of a school. The height, h, in metres, after t seconds can be approximated by $h = 15 + 22t - 5t^2$.

a) What is the height of the school?

b) How long does it take for the rocket to pass a window that is 10 m above the ground?

c) When does the rocket hit the ground?

d) What is the maximum height of the rocket?

PRACTISING

Safety Connection

Smoke detectors provide early warning of fire or smoke. Monitored smoke detectors send a signal to a monitoring station.

3. Water from a fire hose is sprayed on a fire that is coming from a window. The window is 15 m up the side of a wall. The equation $H = -0.011x^2 + 0.99x + 1.6$ models the height of the jet of water, H, and the horizontal distance it can travel from the nozzle, x, both in metres.

a) What is the maximum height that the water can reach?

b) How far back could a firefighter stand, but still have the water reach the window?

4. Brett is jumping on a trampoline in his backyard. Each jump takes about 2 s from beginning to end. He passes his bedroom window, which is 4 m high, 0.4 s into each jump. By modelling Brett's height with a quadratic relation, determine his maximum height.

5. Pauline wants to sell stainless steel water bottles as a school fundraiser. She knows that she will maximize profits, and raise \$1024, if she sells the bottles for \$28 each. She also knows that she will lose \$4160 if she sells the bottles for only \$10 each.

a) Write a quadratic relation to model her profit, P, in dollars, if she sells the bottles for x dollars each.

b) What selling price will ensure that she breaks even?

6. K A professional stunt performer at a theme park dives off a tower, which is 21 m high, into water below. The performer's height, h, in metres, above the water at t seconds after starting the jump is given by $h = -4.9t^2 + 21$.

a) How long does the performer take to reach the halfway point?

b) How long does the performer take to reach the water?

c) Compare the times for parts a) and b). Explain why the time at the bottom is not twice the time at the halfway point.

7. A Harold wants to build five identical pigpens, side by side, on his farm using 30 m of fencing. Determine the dimensions of the enclosure that would give his pigs the largest possible area. Calculate this area.

8. A biologist predicts that the deer population, P, in a certain national park can be modelled by $P = 8x^2 - 112x + 570$, where x is the number of years since 1999.

a) According to this model, how many deer were in the park in 1999?

b) In which year was the deer population a minimum? How many deer were in the park when their population was a minimum?

c) Will the deer population ever reach zero, according to this model?

d) Would you use this model to predict the number of deer in the park in 2020? Explain.

9. The depth underwater, d, in metres, of Daisy the dolphin during a dive can be modelled by $d = 0.1t^2 - 3.5t + 6$, where t is the time in seconds after the dolphin begins her descent toward the water.
a) How long was Daisy underwater?
b) How deep did Daisy dive?

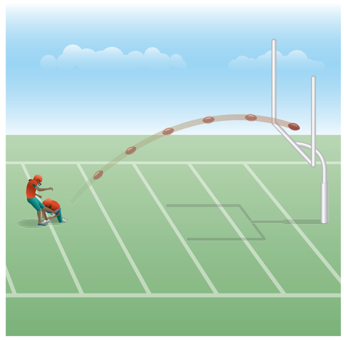
10. The cost, C, in dollars per hour, to run a machine can be modelled by $C = 0.01x^2 - 1.5x + 93.25$, where x is the number of items produced per hour.
a) How many items should be produced each hour to minimize the cost?
b) What production rate will keep the cost below \$53?

11. Nick has a beautiful rectangular garden, which measures 3 m by 3 m. He wants to create a uniform border of river rocks around three sides of his garden. If he wants the area of the border and the area of his garden to be equal, how wide should the border be?

12. A ball was thrown from the top of a playground jungle gym, which is 1.5 m high. The ball reached a maximum height of 4.2 m when it was 3 m from where it was thrown. How far from the jungle gym was the ball when it hit the ground?

13. The sum of the squares of three consecutive even integers is 980.
T Determine the integers.

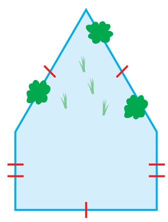
14. Maggie can kick a football 34 m, reaching a maximum height of 16 m.
a) Write an equation to model this situation.
b) To score a field goal, the ball has to pass between the vertical poles and over the horizontal bar, which is 3.3 m above the ground. How far away from the uprights can Maggie be standing so that she has a chance to score a field goal?



15. **a)** Create a problem that you could model using a quadratic relation
C and you could solve using the corresponding quadratic equation.
b) Create a problem that you could model using a quadratic relation and you could solve by determining the coordinates of the vertex.

Extending

16. Mark is designing a pentagonal-shaped play area for a daycare facility. He has 30 m of nylon mesh to enclose the play area. The triangle in the diagram is equilateral. Calculate the dimensions of the rectangle and the triangle, to the nearest tenth of a metre, that will maximize the area he can enclose for the play area.



17. Richie walked 15 m diagonally across a rectangular field. He then returned to his starting position along the outside of the field. The total distance he walked was 36 m. What are the dimensions of the field?