

6.1

Solving Quadratic Equations

YOU WILL NEED

- grid paper
- ruler
- graphing calculator



quadratic equation

an equation that contains at least one term whose highest degree is 2; for example, $x^2 + x - 2 = 0$

root

a solution; a number that can be substituted for the variable to make the equation a true statement; for example, $x = 1$ is a root of $x^2 + x - 2 = 0$, since $1^2 + 1 - 2 = 0$

GOAL

Use graphical and algebraic strategies to solve quadratic equations.

INVESTIGATE the Math

Andy and Susie run a custom T-shirt business. From past experience, they know that they can model their expected profit, in dollars, with the relation $P = -x^2 + 120x - 2000$, where x is the number of T-shirts they sell. Andy wants to sell enough T-shirts to earn \$1200. Susie wants to sell just enough T-shirts to break even because she wants to close the business.

? How can Andy and Susie determine the number of T-shirts they must sell to achieve their goals?

- Why can you use the **quadratic equation** $-x^2 + 120x - 2000 = 0$ to determine the number of T-shirts that must be sold to achieve Susie's goal?
- Factor the left side of the equation in part A. Use the factors to determine the number of T-shirts that must be sold to achieve Susie's goal.
- Use your factors for part B to predict what the graph of the profit relation will look like. Sketch the graph, based on your prediction.
- Graph the profit relation using a graphing calculator. Was your prediction for part C correct?
- What quadratic equation can you use to describe Andy's goal of making a profit of \$1200?
- How can you use your graph for part D to determine the **roots** of your equation for part E?
- How many T-shirts must be sold to achieve Andy's goal?

Reflecting

- Why did factoring $-x^2 + 120x - 2000$ help you determine the break-even points?
- Are the roots of the equation $-x^2 + 120x - 2000 = 0$ also zeros or x -intercepts of the relation $y = -x^2 + 120x - 2000$? Explain.

- J. Why would factoring the left side of $-x^2 + 120x - 2000 = 1200$ not help you determine the number of T-shirts that Andy has to sell?
- K. Explain why it would help you solve the equation in part J if you were to write it as $-x^2 + 120x - 2000 - 1200 = 0$.
- L. To solve $ax + b = c$, you isolate x . Why would you not isolate x^2 to solve $ax^2 + bx + c = 0$?

APPLY the Math

EXAMPLE 1

Selecting a strategy to solve a quadratic equation

The user's manual for Arleen's model rocket says that the equation $h = -5t^2 + 40t$ models the approximate height, in metres, of the rocket after t seconds. When will Arleen's rocket reach a height of 60 m?

Amir's Solution: Selecting a factoring strategy

$$-5t^2 + 40t = 60$$

I substituted 60 for h because I wanted to calculate the time for the height 60 m.

$$-5t^2 + 40t - 60 = 0$$

I subtracted 60 from both sides of the equation to make the right side equal zero. I did this so that I could determine the zeros of the corresponding relation.

$$-5(t^2 - 8t + 12) = 0$$

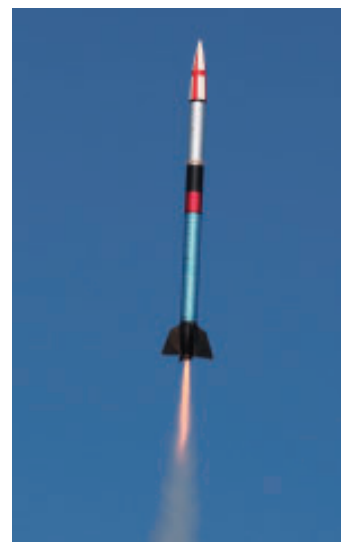
$$-5(t - 2)(t - 6) = 0$$

$$t - 2 = 0 \text{ or } t - 6 = 0$$

$$t = 2 \qquad t = 6$$

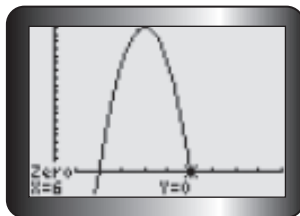
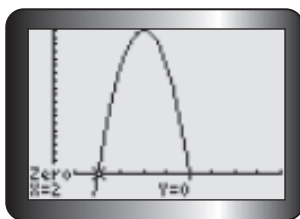
I divided out the common factor of -5 . Then I factored the trinomial. The trinomial will equal zero if either factor equals zero. I set each factor equal to zero and solved both equations. This gave me the zeros of the parabola.

The rocket is 60 m above the ground at 2 s on the way up and 6 s on the way down.



Tech Support

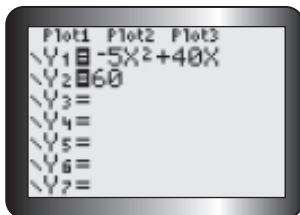
For help locating the zeros of a relation using a TI 83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.



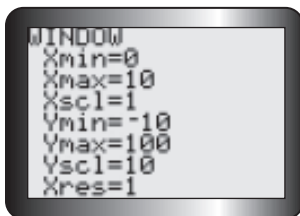
My solutions were correct.

I verified my solutions by graphing $y = -5x^2 + 40x - 60$, and then locating the zeros.

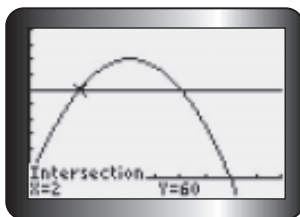
Alex's Solution: Selecting a graphing strategy



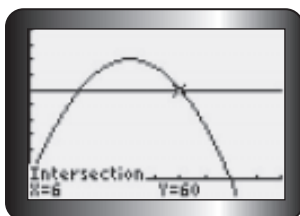
Using a graphing calculator, I entered the height equation in Y1. I substituted the variable y for h and the variable x for t . I entered 60 in Y2 to determine when the rocket will reach 60 m.



I estimated that the rocket will travel about 100 m and be in the air for about 10 s, so I used these window settings.



I used the Intersect operation to locate the intersection points of the two graphs.



The rocket is 60 m off the ground after 2 s and after 6 s.

Tech Support

For help determining points of intersection using a TI-83/84 graphing calculator, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.

EXAMPLE 2 **Selecting a factoring strategy to solve a quadratic equation**

Determine the roots of $6x^2 - 11x - 10 = 0$.

Annette's Solution

$$6x^2 - 11x - 10 = 0$$

$$\text{Product} = -60 \quad \text{Sum} = -11$$

$$(1)(-60) \quad 1 + (-60) = -59 \times$$

$$(2)(-30) \quad 2 + (-30) = -28 \times$$

$$(3)(-20) \quad 3 + (-20) = -17 \times$$

$$(4)(-15) \quad 4 + (-15) = -11 \checkmark$$

Since the trinomial in the equation contains no common factors and is one where $a \neq 1$, I used decomposition. I looked for two numbers whose sum is -11 and whose product is $(6)(-10) = -60$.

$$6x^2 - 15x + 4x - 10 = 0$$

$$3x(2x - 5) + 2(2x - 5) = 0$$

$$(2x - 5)(3x + 2) = 0$$

Since the numbers were -15 and 4 , I used these to decompose the middle term. I factored the first two terms and then the last two terms. Then, I divided out the common factor of $2x - 5$.

$$2x - 5 = 0 \quad \text{or} \quad 3x + 2 = 0$$

$$2x = 5 \quad 3x = -2$$

$$x = \frac{5}{2} \quad x = -\frac{2}{3}$$

I set each factor equal to zero and solved each equation.

The roots of $6x^2 - 11x - 10 = 0$

$$\text{are } x = 2\frac{1}{2} \text{ and } x = -\frac{2}{3}.$$

EXAMPLE 3 **Reasoning about how to solve a quadratic equation**

Determine all the values of x that satisfy the equation $x^2 + 4 = 3x(x - 5)$.

If necessary, round your answers to two decimal places.

Karl's Solution

$$x^2 + 4 = 3x(x - 5)$$

$$x^2 + 4 = 3x^2 - 15x$$

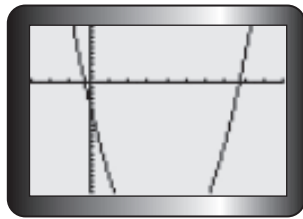
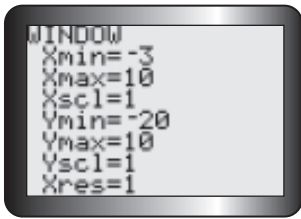
I decided to write an equivalent equation in the form $ax^2 + bx + c = 0$, which I could solve by graphing or factoring. I expanded the expression on the right side of the equation.

$$0 = 3x^2 - x^2 - 15x - 4$$

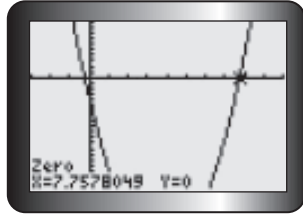
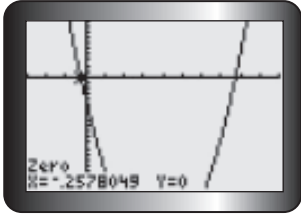
$$0 = 2x^2 - 15x - 4$$

I used inverse operations to make the left side of the equation equal to zero. I couldn't factor the right side of the equation, so I decided to use a graph.





I graphed $y = 2x^2 - 15x - 4$ using these window settings. From the graph, I could see that one x -intercept was between -1 and 0 and the other x -intercept was between 7 and 8 .



Using the Zero operation of the calculator, I estimated that the x -intercepts were about -0.258 and 7.758 .

Tech Support

For help determining the zeros of a relation using a TI-83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.

The solutions are $x \doteq -0.26$ and $x \doteq 7.76$.

I rounded the solutions to two decimal places. These are reasonable estimates, since the solutions are not exact.

EXAMPLE 4 Reflecting on the reasonableness of a solution

A ball is thrown from the top of a seaside cliff. Its height, h , in metres, above the sea after t seconds can be modelled by $h = -5t^2 + 21t + 120$. How long will the ball take to fall 20 m below its initial height?

Jacqueline's Solution

$$h = -5t^2 + 21t + 120$$

$$h = -5(0)^2 + 21(0) + 120$$

$$h = 120$$

The cliff is 120 m high, so the ball starts 120 m above the sea.

$$120 - 20 = 100$$

Let $h = 100$.

$$100 = -5t^2 + 21t + 120$$

$$0 = -5t^2 + 21t + 120 - 100$$

$$0 = -5t^2 + 21t + 20$$

I let $t = 0$ to determine the initial height of the ball.

The initial height of the ball was 120 m. When the ball had fallen 20 m below its initial height, it was 100 m above the sea.

I substituted 100 for h in the relation. I wrote the equation in the form $0 = ax^2 + bx + c$ so that I could solve it by graphing or factoring.

I subtracted 100 from both sides of the equation to make the left side equal to 0.



$$0 = 5t^2 - 21t - 20$$

$$0 = 5t^2 - 25t + 4t - 20$$

$$0 = 5t(t - 5) + 4(t - 5)$$

$$0 = (5t + 4)(t - 5)$$

I multiplied all the terms, on both sides of the equation, by -1 because I wanted $5t^2$ to be positive. I factored the right side of the equation using decomposition.

$$5t + 4 = 0 \quad \text{or} \quad t - 5 = 0$$

$$5t = -4 \quad t = 5$$

$$t = -\frac{4}{5}$$

I set each factor equal to zero and solved for t .

The ball will take 5 s to fall 20 m below its initial height.

Since the ball was thrown at $t = 0$, I knew that the solution $t = -\frac{4}{5}$ didn't make sense. I used the solution $t = 5$ since this did make sense.

In Summary

Key Ideas

- A quadratic equation is any equation that contains a polynomial in one variable whose degree is 2; for example, $x^2 + 6x + 9 = 0$.
- All quadratic equations can be expressed in the form $ax^2 + bx + c = 0$ using algebraic strategies. In this form, the equation can be solved by
 - factoring the quadratic expression, setting each factor equal to zero, and solving the resulting equations
 - or
 - graphing the corresponding relation $y = ax^2 + bx + c$ and determining the zeros, or x -intercepts

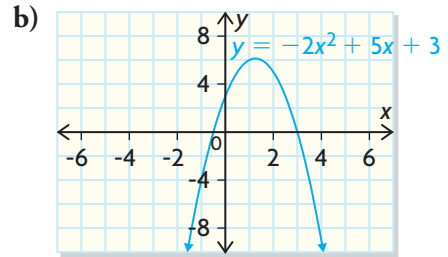
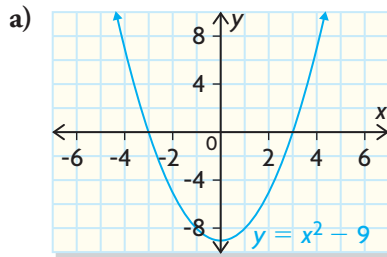
Need to Know

- Roots and solutions have the same meaning. These are all values that satisfy an equation.
- Some quadratic equations can be solved by factoring. Other quadratic equations must be solved by using a graph.
- If you use factoring to solve a quadratic equation, write the equation in the form $ax^2 + bx + c = 0$ before you try to factor.
- To solve $ax^2 + bx + c = d$ using a graph, graph $y = ax^2 + bx + c$ and $y = d$ on the same axes. The solutions to the equation are the x -coordinates of the points where the parabola and the horizontal line intersect.

CHECK Your Understanding

1. The solutions to each equation are the x -intercepts of the corresponding quadratic relation. State the quadratic relation.
 - a) $x^2 - 4x + 4 = 0$
 - b) $2x^2 - 9x = 5$

2. Use the graph of each quadratic relation to determine the roots to each quadratic equation, where $y = 0$.



3. Solve each equation.

a) $x(x + 4) = 0$

d) $(3x + 8)(x - 4) = 0$

b) $(x + 10)(x + 8) = 0$

e) $x^2 + 5x + 6 = 0$

c) $(x - 5)^2 = 0$

f) $x^2 - 2x = 8$

PRACTISING

4. Determine whether the given value is a root of the equation.

a) $x = 2; x^2 + x - 6 = 0$

d) $x = \frac{3}{2}; 8x^2 + 10x - 3 = 0$

b) $x = 4; x^2 + 7x - 8 = 0$

e) $x = -5; x^2 - 4x - 5 = 0$

c) $x = -\frac{1}{2}; 2x^2 + 11x + 5 = 0$

f) $x = 2; 3x^2 - 2x - 8 = 0$

5. Solve each equation by factoring. Use an equivalent equation, if necessary.

a) $x^2 + 2x - 15 = 0$

d) $x^2 - 5x = 0$

b) $x^2 + 5x - 24 = 0$

e) $x^2 - 6x = 16$

c) $x^2 + 4x + 4 = 0$

f) $x^2 + 12 = 7x$

6. Solve by factoring. Verify your solutions.

a) $3x^2 - 5x - 2 = 0$

d) $6x^2 - x - 2 = 0$

b) $2x^2 + 3x - 2 = 0$

e) $4x^2 - 4x = 3$

c) $3x^2 - 4x - 15 = 0$

f) $9x^2 + 1 = 6x$

7. Simplify and then solve each equation.

K a) $x(x + 1) = 12$

d) $3x(x + 6) + 50 = 2x^2 + 3(x - 2)$

b) $2x(x + 4) = x + 4$

e) $(x + 2)^2 + x = 2(3x + 5)$

c) $3x(x + 2) = 2x^2 - (4 - x)$

f) $(2x + 1)^2 = x + 2$

8. Determine the roots of each equation.

a) $x^2 + 4x - 32 = 0$

d) $x^2 + 5x = 14$

b) $x^2 + 11x + 30 = 0$

e) $4x^2 + 25 = 20x$

c) $5x^2 - 28x - 12 = 0$

f) $3x^2 + 16x - 7 = 5$

9. Solve each equation. Round your answers to two decimal places.

a) $x^2 + 5x - 2 = 0$

d) $x(x + 5) = 2x + 7$

b) $4x^2 - 8x + 3 = 0$

e) $3x^2 + 5x - 3 = x^2 + 4x + 1$

c) $x^2 + 1 = 4 - 2x^2$

f) $(x + 3)^2 - 2x = 15$

10. Conor has a summer lawn-mowing business. Based on experience, Conor knows that $P = -5x^2 + 200x - 1500$ models his profit, P , in dollars, where x is the amount, in dollars, charged per lawn.
- How much does he need to charge if he wants to break even?
 - How much does he need to charge if he wants to have a profit of \$500?
11. Stacey maintains the gardens in the city parks. In the summer, she plans to build a walkway through the rose garden. The area of the walkway, A , in square metres, is given by $A = 160x + 4x^2$, where x is the width of the walkway in metres. If the area of the walkway must be 900 m^2 , determine the width.
12. Patrick owns an apartment building. He knows that the money he earns in a month depends on the rent he charges. This relationship can be modelled by $E = \frac{1}{50}R(1650 - R)$, where E is Patrick's monthly earnings, in dollars, and R is the amount of rent, in dollars, he charges each tenant.
- How much will he earn if he sets the rent at \$900?
 - If Patrick wants to earn at least \$13 000, between what two values should he set the rent?
13. Determine the points of intersection of the line $y = -2x + 7$ and **T** the parabola $y = 2x^2 + 3x - 5$.
14. While hiking along the top of a cliff, Harlan knocked a pebble over **A** the edge. The height, h , in metres, of the pebble above the ground after t seconds is modelled by $h = -5t^2 - 4t + 120$.
- How long will the pebble take to hit the ground?
 - For how long is the height of the pebble greater than 95 m?
15. Is it possible to solve a quadratic equation that is not factorable over **C** the set of integers? Explain.
16. **a)** Describe when and why you would rewrite a quadratic equation to solve it. In your answer, include $x^2 - 2x = 15$, rewritten as $x^2 - 2x - 15 = 0$.
- b)** Explain how the relation $y = x^2 - 2x - 15$ can be used to solve $x^2 - 2x - 15 = 0$.



Environment Connection

By photosynthesis, green plants remove carbon dioxide from the air and produce oxygen.



Extending

17. Solve the equations $x^4 - 9x^2 + 20 = 0$ and $x^3 - 9x^2 + 20x = 0$ by first solving the equation $x^2 - 9x + 20 = 0$.
18. Will all quadratic equations always have two solutions? Explain how you know and support your claim with examples.