5.6 Connecting Standard and Vertex Forms

GOAL

Sketch or graph a quadratic relation with an equation of the form $y = ax^2 + bx + c$ using symmetry.

YOU WILL NEED

- grid paper
- ruler

INVESTIGATE the Math

Many places hold a fireworks display on Canada Day. Clayton, a member of the local fire department, launches a series of rockets from a barge that is floating in the middle of the lake. Each rocket is choreographed to explode at the correct time. The equation $h = -5t^2 + 40t + 2$ can be used to model the height, h, of each rocket in metres above the water at t seconds after its launch. A certain rocket is scheduled to explode 3 min 21 s into the program.

Safety *Connection*

Fireworks can cause serious injury when handled incorrectly. It is safer to watch a community display than to create your own.

? Assuming that the rocket will explode at its highest point, when should Clayton launch it from the barge so it will explode at the correct time?

A. What information do you need to determine so that you can model the height of the rocket?

B. Copy and complete the table of values at the right for the rocket. Then plot the points, and sketch the graph of this relation.

Time (s)	Height (m)
0	2
1	37
2	
3	
4	
5	
6	
7	
8	
9	

C. What happens to the rocket between 8 s and 9 s after it is launched?

D. The axis of symmetry of a quadratic relation can be determined from the zeros. In this problem, however, there is only one zero because $t > 0$. Suggest another way to determine the axis of symmetry.

E. The rocket is 2 m above the water when it is launched. When will the rocket be at the same height again? Write the coordinates of these two points.

F. Consider the coordinates of the two points for part E. Why must the axis of symmetry be the same distance from both of these points? What is the equation of the axis of symmetry?

G. How does knowing the equation of the axis of symmetry help you determine the vertex of a parabola? What is the vertex of this parabola?

H. When should Clayton launch the rocket to ensure that it explodes 3 min 21 s into the program?

Reflecting

I. Can you determine the maximum height of the rocket directly from the standard form of the quadratic relation $h = -5t^2 + 40t + 2$? Explain.

J. How did you determine the vertex, even though one of the zeros of the quadratic relation was unknown?

K. Write the quadratic relation in vertex form. How can you compare this equation with the equation given in standard form to determine whether they are identical?

APPLY the Math

EXAMPLE 1 **Connecting** the vertex form to partial factors of the equation

Determine the maximum value of the quadratic relation $y = -3x^2 + 12x + 29$.

Michelle's Solution

$y = -3x^2 + 12x + 29$

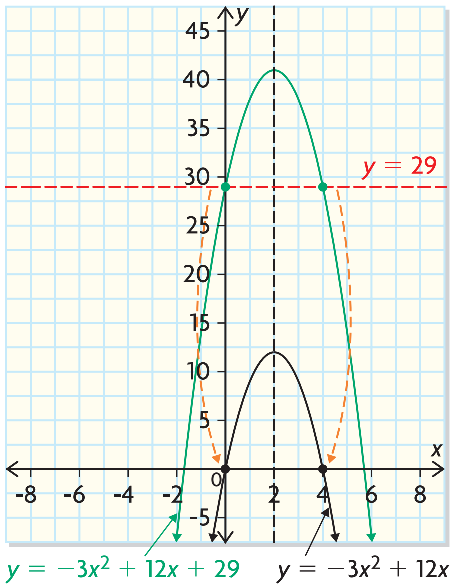
I tried to factor the expression, but I couldn't determine two integers with a product of $(-3) \times 29$ and a sum of 12. This means that I can't use the zeros to help me.

$y = x(-3x + 12) + 29$

I had to determine the axis of symmetry, since the vertex (where the maximum value occurs) lies on it. To do this, I had to locate two points with the same *y*-coordinate. I removed a partial factor of *x* from the first two terms.

When $y = 29$,
$29 = x(-3x + 12) + 29$
$x = 0$ or $-3x + 12 = 0$
$x = 0 \qquad x = 4$

I noticed that the *y*-value will be 29 if either factor in the equation equals 0. I decided to determine the two points on the parabola with a *y*-coordinate of 29 by setting each partial factor equal to 0 and solving for *x*.



I knew that this would work because it was like translating the graph down 29 units to make the points with a y-coordinate of 29 turn into points with a y-coordinate of 0.

This let me determine the zeros of the new translated graph.

I noticed that the axis of symmetry was the same for the two graphs.

(0, 29) and (4, 29) are on the graph of the original quadratic relation. This makes sense since the points (0, 0) and (4, 0) are the zeros of the translated graph.

These points are the same distance from the axis of symmetry. So, I know that the axis of symmetry is halfway between $x = 0$ and $x = 4$.

The equation of the axis of symmetry is

$$x = \frac{0 + 4}{2} \text{ so } x = 2.$$

I calculated the mean of the x-coordinates of these points to determine the axis of symmetry.

$$y = -3(2)^2 + 12(2) + 29$$
$$y = -12 + 24 + 29$$
$$y = 41$$

The y-coordinate of the vertex is the maximum value because the graph opens downward. To determine the maximum, I substituted $x = 2$ into $y = -3x^2 + 12x + 29$.

The maximum value is 41.

EXAMPLE 2 Selecting a partial factoring strategy to sketch the graph of a quadratic relation

Express the quadratic relation $y = 2x^2 + 8x + 5$ in vertex form. Then sketch a graph of the relation by hand.

Marnie's Solution

$$y = 2x^2 + 8x + 5$$
$$y = x(2x + 8) + 5$$

This equation cannot be factored fully since you can't determine two integers with a product of 2×5 and a sum of 8. I removed a partial factor of x from the first two terms.

$$x = 0 \text{ or } 2x + 8 = 0$$
$$x = 0 \qquad x = -4$$

I found two points with a y-coordinate of 5 by setting each partial factor equal to 0. Both of these points are the same distance from the axis of symmetry.

The points (0, 5) and (−4, 5) are on the parabola.

The axis of symmetry is $x = -2$.

I found the axis of symmetry by calculating the mean of the x-coordinates of these points.

At the vertex,

$y = 2(-2)^2 + 8(-2) + 5$

$y = -3$

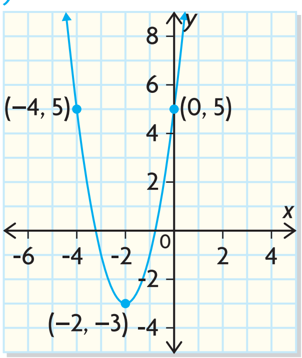
Since the parabola is symmetric, the vertex is on the line $x = -2$. I substituted this value into the relation.

The vertex of the parabola is at $(-2, -3)$.

In vertex form, the equation of the parabola is $y = 2(x + 2)^2 - 3$.

I know that the value of a is the same in standard form and vertex form. In this case, $a = 2$.

$y = 2x^2 + 8x + 5$



The parameter a is positive, so the parabola opens upward.

I used the vertex and the two points I found, (0, 5) and (−4, 5), to sketch the parabola.

In Summary

Key Idea

- If a quadratic relation is in standard form and cannot be factored fully, you can use partial factoring to help you determine the axis of symmetry of the parabola. Then you can use the axis of symmetry to determine the coordinates of the vertex.

Need to Know

- If $y = ax^2 + bx + c$ cannot be factored, you can express the relation in the partially factored form $y = x(ax + b) + c$. Then you can use this form to determine the vertex form:
  - Set $x(ax + b) = 0$ and solve for x to determine two points on the parabola that are the same distance from the axis of symmetry. Both of these points have y-coordinate c.
  - Determine the axis of symmetry, $x = h$, by calculating the mean of the x-coordinates of these points.
  - Substitute $x = h$ into the relation to determine k, the y-coordinate of the vertex.
  - Substitute the values of a, h, and k into $y = a(x - h)^2 + k$.

CHECK Your Understanding

1. Determine the equation of the axis of symmetry of a parabola that passes through points (2, 8) and (−6, 8).

2. Determine two points that are the same distance from the axis of symmetry of the quadratic relation $y = 4x^2 - 12x + 5$.

3. Use partial factoring to determine the vertex form of the quadratic relation $y = 2x^2 - 10x + 11$.

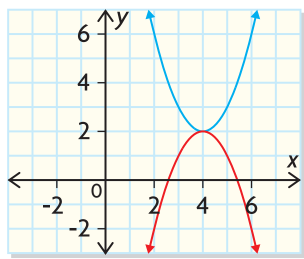
PRACTISING

4. A parabola passes through points (3, 0), (7, 0), and (9, −24).
   a) Determine the equation of the axis of symmetry.
   b) Determine the coordinates of the vertex, and write the equation in vertex form.
   c) Write the equation in standard form.

5. For each quadratic relation,
   i) determine the coordinates of two points on the graph that are the same distance from the axis of symmetry
   ii) determine the equation of the axis of symmetry
   iii) determine the coordinates of the vertex
   iv) write the relation in vertex form

   a) $y = (x - 1)(x + 7)$
   b) $y = x(x - 6) - 8$
   c) $y = -2(x + 3)(x - 7)$
   d) $y = x(3x + 12) + 2$
   e) $y = x^2 + 5x$
   f) $y = x^2 - 11x + 21$

6. The equation of one of these parabolas at the right is $y = x^2 - 8x + 18$.
   K Determine the equation of the other in vertex form.



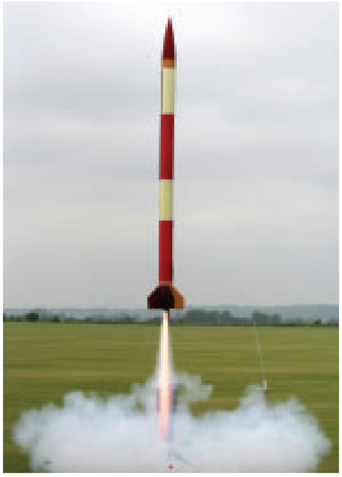
7. For each quadratic relation,
   i) use partial factoring to determine two points that are the same distance from the axis of symmetry
   ii) determine the coordinates of the vertex
   iii) express the relation in vertex form
   iv) sketch the graph

   a) $y = x^2 - 6x + 5$
   b) $y = x^2 - 4x - 11$
   c) $y = -2x^2 + 12x - 11$
   d) $y = -x^2 - 6x - 13$
   e) $y = -\frac{1}{2}x^2 + 2x - 3$
   f) $y = 2x^2 - 10x + 11$

8. Use two different strategies to determine the equation of the axis
   C of symmetry of the parabola defined by $y = -2x^2 + 16x - 24$. Which strategy do you prefer? Explain why.

9. Determine the values of a and b in the relation $y = ax^2 + bx + 7$ if the vertex is located at $(4, -5)$.
T

10. Determine the values of a and b in the relation $y = ax^2 + bx + 8$ if the vertex is located at $(1, 7)$.



11. A model rocket is launched straight up, with an initial velocity of 150 m/s. The height of the rocket can be modelled by $h = -5t^2 + 150t$, where h is the height in metres and t is the elapsed time in seconds. What is the maximum height reached by the rocket?

12. A baseball is hit from a height of 1 m. The height, h, of the ball in metres after t seconds can be modelled by $h = -5t^2 + 9t + 1$. Determine the maximum height reached by the ball.
A

13. A movie theatre can accommodate a maximum of 450 moviegoers per day. The theatre operators have determined that the profit per day, P, is related to the ticket price, t, by $P = -30t^2 + 450t - 790$. What ticket price will maximize the daily profit?

14. The world production of gold from 1970 to 1990 can be modelled by $G = 1492 - 76t + 5.2t^2$, where G is the number of tonnes of gold and t is the number of years since 1970 ($t = 0$ for 1970, $t = 1$ for 1971, and so on).
    - **a)** During this period, when was the minimum amount of gold mined?
    - **b)** What was the least amount of gold mined in one year?
    - **c)** How much gold was mined in 1985?

15. Create a concept web that summarizes the different algebraic strategies you can use to determine the axis of symmetry and the vertex of a quadratic relation given in the form $y = ax^2 + bx + c$.

Extending

16. A farmer has $3000 to spend on fencing for two adjoining rectangular pastures, both with the same dimensions. A local contracting company can build the fence for $5.00/m. What is the largest total area that the farmer can have fenced for this price?

17. A city transit system carries an average of 9450 people per day on its buses, at a fare of $1.75 each. The city wants to maximize the transit system's revenue by increasing the fare. A survey shows that the number of riders will decrease by 210 for every $0.05 increase in the fare. What fare will result in the greatest revenue? How many daily riders will they lose at this new fare?