5.4 Quadratic Models Using Vertex Form

YOU WILL NEED

- grid paper
- ruler
- graphing calculator
- spreadsheet program (optional)

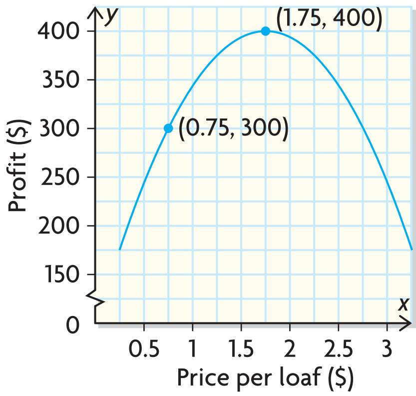
GOAL

Write the equation of the graph of a quadratic relation in vertex form.

LEARN ABOUT the Math

The Best Bread Bakery wants to determine its daily profit from bread sales. This graph shows the data gathered by the company.

Bakery Profits from Bread Sales



? What equation represents the relationship between the price of bread and the daily profit from bread sales?

EXAMPLE 1 **Connecting** a parabola to the vertex form of its equation

Determine the equation of this quadratic relation from its graph.

Sabrina's Solution

$y = a(x - h)^2 + k$

Since the graph is a parabola and the coordinates of the vertex are given, I decided to use vertex form.

$y = a(x - 1.75)^2 + 400$

$300 = a(0.75 - 1.75)^2 + 400$

$300 = a(-1)^2 + 400$

$300 = a + 400$

$-100 = a$

Since (1.75, 400) is the vertex, $h = 1.75$ and $k = 400$. I substituted these values into the equation.

To determine the value of a, I chose the point (0.75, 300) on the graph. I substituted these coordinates for x and y in the equation.

I followed the order of operations and solved for the value of a.

The equation that represents the relationship is $y = -100(x - 1.75)^2 + 400$.

Reflecting

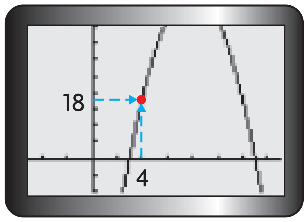
A. What information do you need from the graph of a quadratic relation to determine the equation of the relation in vertex form?

B. You have used the standard, factored, and vertex forms of a quadratic relation. Which form do you think is most useful for determining the equation of a parabola from its graph? Explain why.

APPLY the Math

EXAMPLE 2 **Connecting** information about a parabola to its equation

The graph of $y = x^2$ was stretched by a factor of 2 and reflected in the x-axis. The graph was then translated to a position where its vertex is not visible in the viewing window of a graphing calculator. Determine the quadratic relation in vertex form from the partial graph displayed in the screen shot. For each tick mark, the scale on the y-axis is 5, and the scale on the x-axis is 2.



Terri's Solution

$a = -2$

$y = -2(x - h)^2 + k$

The graph was stretched by a factor of 2 and reflected in the x-axis.

I substituted the value of a into the vertex form of the quadratic relation.

The zeros of the graph are 3 and 13.

$$h = \frac{3 + 13}{2}$$

$$h = 8$$

I determined the mean of the two zeros to calculate the value of h. The vertex lies on the axis of symmetry, which is halfway between the zeros of the graph.

$$18 = -2(4 - 8)^2 + k$$
$$18 = -2(16) + k$$
$$18 = -32 + k$$
$$50 = k$$

I saw that (4, 18) is a point on the graph. By substituting these coordinates, as well as the value I determined for h, I was able to solve for k.

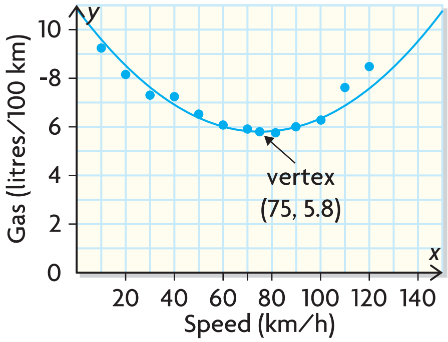
The equation of the graph is $y = -2(x - 8)^2 + 50$.

EXAMPLE 3 **Selecting a strategy** to determine a quadratic model

The amount of gasoline that a car consumes depends on its speed. A group of students decided to research the relationship between speed and fuel consumption for a particular car. They collected the data in the table. Determine an equation that models the relationship between speed and fuel consumption.

Speed (km/h)	10	20	30	40	50	60	70	80	90	100	110	120
Gas Consumed (litres/100 km)	9.2	8.1	7.4	7.2	6.4	6.1	5.9	5.8	6.0	6.3	7.5	8.4

Eric's Solution: Representing a relation with a scatter plot and determining the equation algebraically



I constructed a scatter plot to display the data and drew a curve of good fit. Since the curve looked parabolic and I knew that I could estimate the coordinates of the vertex. I estimated the coordinates of the vertex to be about (75, 5.8).

$$y = a(x - h)^2 + k$$
$$y = a(x - 75)^2 + 5.8$$

I decided to use the vertex form of the equation. I substituted the estimated values (75, 5.8) into the general equation.

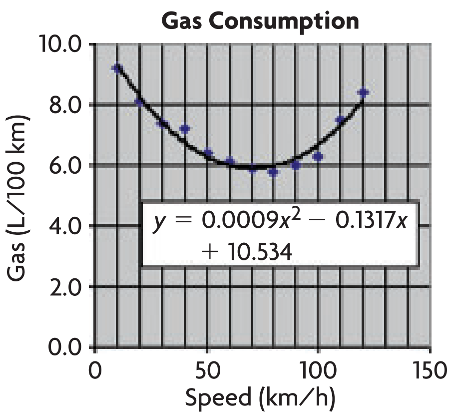
$$6.0 = a(90 - 75)^2 + 5.8$$

From the table, I knew that the point (90, 6.0) is close to the curve. I substituted the coordinates of this point for *x* and *y* to determine *a*.

$$6.0 = a(15)^2 + 5.8$$
$$6.0 = 225a + 5.8$$
$$0.2 = 225a$$
$$0.0009 \doteq a$$

I solved for *a*.

The equation that models the data is $y = 0.0009(x - 75)^2 + 5.8$.



I checked my equation using a spreadsheet. I entered the data from the table. I used column A for the *Speed* values and column B for the *Gas* values. I created a graph, added a trend line using **quadratic regression** of order 2, and chose the option to display the equation on the graph.

Tech | Support

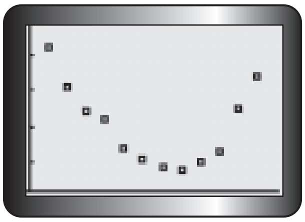
For help creating a scatter plot and performing a regression analysis using a spreadsheet, see Appendix B-35.

$y = 0.0009(x - 75)^2 + 5.8$
$y = 0.0009(x^2 - 150x + 5625) + 5.8$
$y = 0.0009x^2 - 0.135x + 5.0625 + 5.8$
$y = 0.0009x^2 - 0.135x + 10.8625$

The spreadsheet equation was in standard form, but my equation was in vertex form. To compare the two equations, I expanded my equation.

The two equations are very close, so they are both good quadratic models for this set of data.

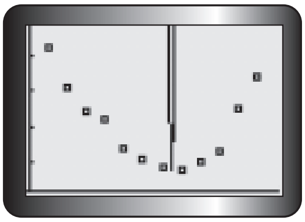
Gillian's Solution: Selecting a graphing calculator and an informal curve-fitting process



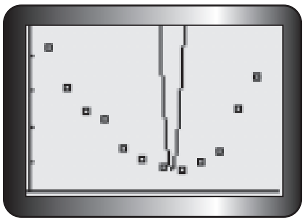
I entered the data into L1 and L2 in the data editor of a graphing calculator and created a scatter plot.

$y = a(x - 75)^2 + 5.8$

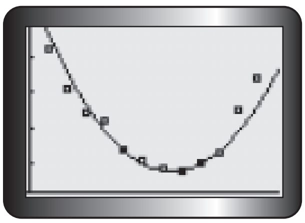
The points had a parabolic pattern, so I estimated the coordinates of the vertex to be about (75, 5.8). I substituted these coordinates into the general equation.



Since the parabola opens upward, I knew that $a > 0$. I used $a = 1$ and entered the equation $y = 1(x - 75)^2 + 5.8$ into Y1 of the equation editor. Then I graphed the equation. The location of the vertex looked good, but the parabola wasn't wide enough.



I decreased the value of a to $a = 0.1$, but the parabola still wasn't wide enough.

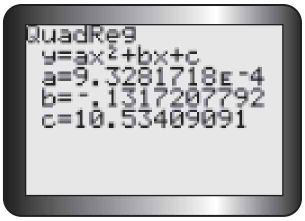


I decreased the value of a several more times until I got a good fit. I found that $a = 0.0009$ worked fairly well.

An equation that models the relationship between speed and fuel consumption is $y = 0.0009(x - 75)^2 + 5.8$.

Tech *Support*

For help creating a scatter plot using a TI-83/84 graphing calculator, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.



I checked my equation by comparing it with the equation produced by quadratic regression on the graphing calculator. To do this, I had to expand my equation.

$$y = 0.0009(x - 75)^2 + 5.8$$
$$y = 0.0009(x^2 - 150x + 5625) + 5.8$$
$$y = 0.0009x^2 - 0.135x + 5.0625 + 5.8$$
$$y = 0.0009x^2 - 0.135x + 10.8625$$

The two equations are close, so they are both good models for this set of data.

Tech *Support*

For help performing a quadratic regression analysis using a TI-83/84 graphing calculator, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.

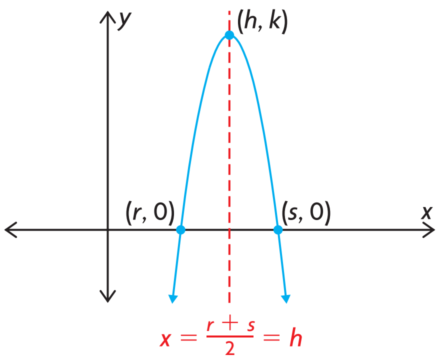
In Summary

Key Idea

- If you know the coordinates of the vertex (h, k) and one other point on a parabola, you can determine the equation of the relation using $y = a(x - h)^2 + k$.

Need to Know

- To determine the value of a, substitute the coordinates of a point on the graph into the general equation and solve for a:
  - If $(h, k) = $ (■, ■), then $y = a(x - $ ■$)^2 + $ ■.
  - If a point on the graph has coordinates $x = $ ■ and $y = $ ■, then, by substitution, ■ $= a($■ $-$ ■$)^2 + $ ■.
  - Since a is the only remaining unknown, its value can be determined by solving the equation.
- The vertex form of an equation can be determined using the zeros of the graph. The axis of symmetry is $x = h$, where h is the mean of the zeros.
- You can convert a quadratic equation from vertex form to standard form by expanding and then collecting like terms.



CHECK Your Understanding

1. Match each equation with the correct graph.

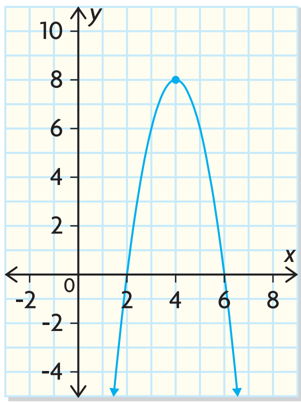
a) $y = 2x^2 - 8$

b) $y = (x + 3)^2$

c) $y = -2(x - 4)^2 + 8$

d) $y = (x - 3)^2 - 8$

i)



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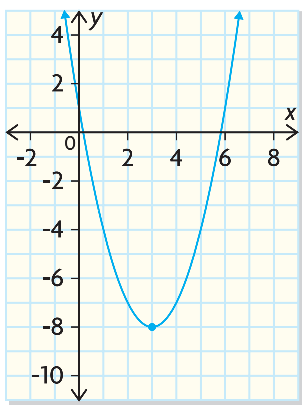

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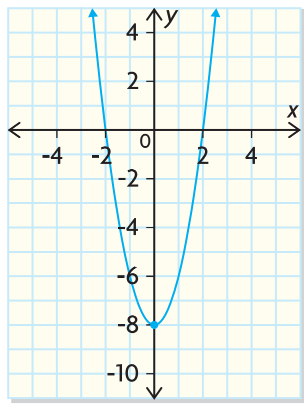

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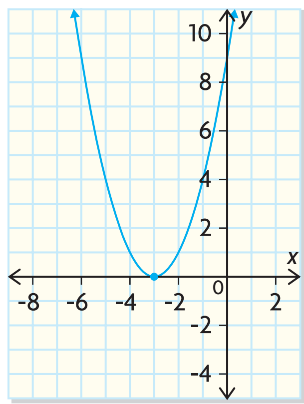


2. The vertex of a quadratic relation is $(4, -12)$.

a) Write an equation to describe all parabolas with this vertex.

b) A parabola with the given vertex passes through point $(13, 15)$. Determine the value of a for this parabola.

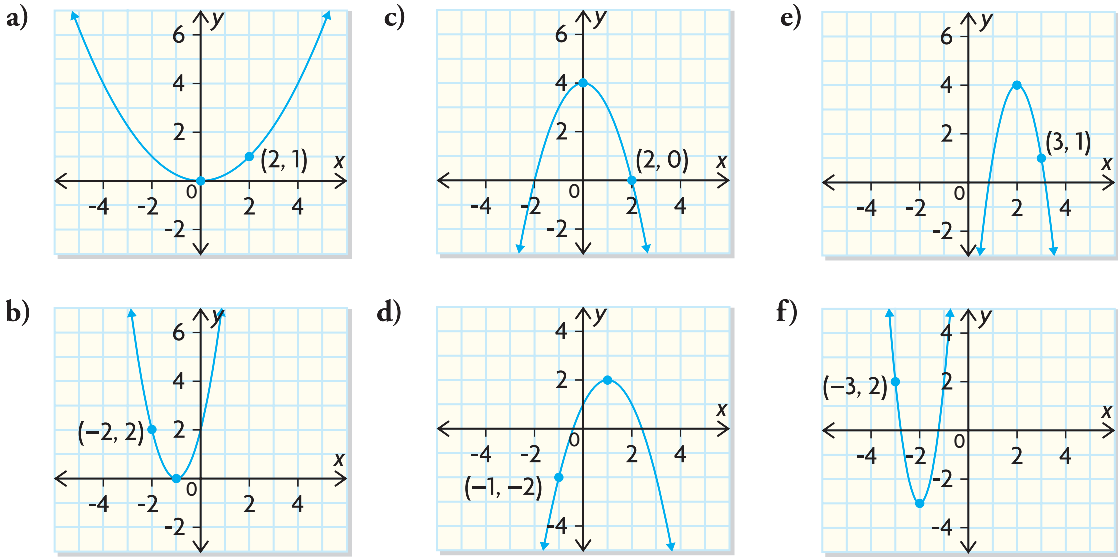
c) Write the equation of the relation for part b).

d) State the transformations that must be applied to $y = x^2$ to obtain the quadratic relation you wrote for part c).

e) Graph the quadratic relation you wrote for part c).

PRACTISING

3. Write the equation of each parabola in vertex form.



4. The following transformations are applied to the graph of $y = x^2$. Determine the equation of each new relation.

a) a vertical stretch by a factor of 4

b) a translation of 3 units left

c) a reflection in the x-axis, followed by a translation 2 units up

d) a vertical compression by a factor of $\frac{1}{2}$

e) a translation of 5 units right and 4 units down

f) a vertical stretch by a factor of 2, followed by a reflection in the x-axis and a translation 1 unit left

5. Write the equation of a parabola with each set of properties.

a) vertex at (0, 4), opens upward, the same shape as $y = x^2$

b) vertex at (5, 0), opens downward, the same shape as $y = x^2$

c) vertex at (2, −3), opens upward, narrower than $y = x^2$

d) vertex at (−3, 5), opens downward, wider than $y = x^2$

e) axis of symmetry $x = 4$, opens upward, two zeros, narrower than $y = x^2$

f) vertex at (3, −4), no zeros, wider than $y = x^2$

6. **K** Determine the equation of a quadratic relation in vertex form, given the following information.

a) vertex at (−2, 3), passes through (−4, 1)

b) vertex at (−1, −1), passes through (0, 1)

c) vertex at (−2, −3), passes through (−5, 6)

d) vertex at (−2, 5), passes through (1, −4)

7. Each table of values defines a parabola. Determine the equation of the axis of symmetry of the parabola, and write the equation in vertex form.

a)

x	y
2	−33
3	−13
4	−1
5	3
6	−1

b)

x	y
0	12
1	4
2	4
3	12
4	28

8. A child kicks a soccer ball so that it barely clears a 2 m fence. The soccer ball lands 3 m from the fence. Determine the equation, in vertex form, of a quadratic relation that models the path of the ball.

9. Data for DVD sales in Canada, over several years, are given in the table.

Year	2002	2003	2004	2005	2006
***x*, Years Since 2002**	0	1	2	3	4
DVDs Sold (1000s)	1446	3697	4573	4228	3702

a) Using graphing technology, create a scatter plot to display the data.
b) Estimate the vertex of the graph you created for part a). Then determine an equation in vertex form to model the data.
c) How many DVDs would you expect to be sold in 2010?
d) Check the accuracy of your model using quadratic regression.

10. A school custodian finds a tennis ball on the roof of the school and
A throws it to the ground below. The table gives the height of the ball above the ground as it moves through the air.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Height (m)	5.00	11.25	15.00	16.25	15.00	11.25	5.00

a) Do the data appear to be linear or quadratic? Explain.
b) Create a scatter plot, and draw a quadratic curve of good fit.
c) Estimate the coordinates of the vertex.
d) Determine an algebraic relation in vertex form to model the data.
e) Use your model to predict the height of the ball at 2.75 s and 1.25 s.
f) How effective is your model for time values that are greater than 3.5 s? Explain.
g) Check the accuracy of your model using quadratic regression.

11. A chain of ice cream stores sells $840 of ice cream cones per day. Each ice cream cone costs $3.50. Market research shows the following trend in revenue as the price of an ice cream cone is reduced.

Price ($)	3.50	3.00	2.50	2.00	1.50	1.00	0.50
Revenue ($)	840	2520	3600	4080	3960	3240	1920

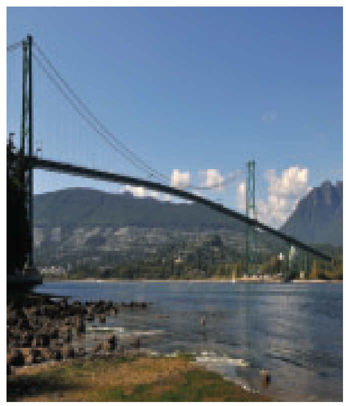
a) Create a scatter plot, and draw a quadratic curve of good fit.
b) Determine an equation in vertex form to model this relation.
c) Use your model to predict the revenue if the price of an ice cream cone is reduced to $2.25.
d) To maximize revenue, what should an ice cream cone cost?
e) Check the accuracy of your model using quadratic regression.

12. This table shows the number of imported cars that were sold in Newfoundland between 2003 and 2007.

Year	2003	2004	2005	2006	2007
Sales of Imported Cars (number sold)	3996	3906	3762	3788	4151

a) Create a scatter plot, and draw a quadratic curve of good fit.
b) Determine an algebraic equation in vertex form to model this relation.
c) Use your model to predict how many imported cars were sold in 2008.
d) What does your model predict for 2006? Is this prediction accurate? Explain.
e) Check the accuracy of your model using quadratic regression.

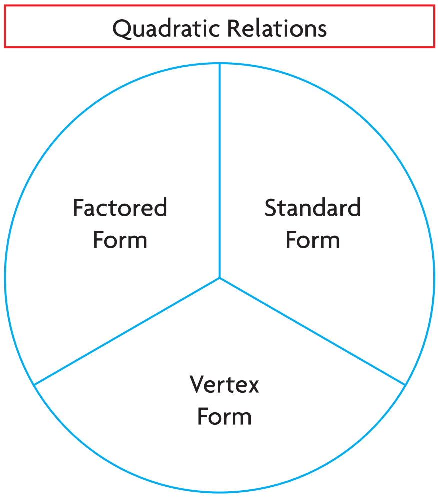
13. **T** The Lion's Gate Bridge in Vancouver, British Columbia, is a suspension bridge that spans a distance of 1516 m. Large cables are attached to the tops of the towers, 50 m above the road. The road is suspended from the large cables by many smaller vertical cables. The smallest vertical cable measures about 2 m. Use this information to determine a quadratic model for the large cables.



14. **C** A model rocket is launched from the ground. After 20 s, the rocket reaches a maximum height of 2000 m. It lands on the ground after 40 s. Explain how you could determine the equation of the relationship between the height of the rocket and time using two different strategies.

15. The owner of a small clothing company wants to create a mathematical model for the company's daily profit, p, in dollars, based on the selling price, d, in dollars, of the dresses made. The owner has noticed that the maximum daily profit the company has made is \$1600. This occurred when the dresses were sold for \$75 each. The owner also noticed that selling the dresses for \$50 resulted in a profit of \$1225. Using a quadratic relation to model this problem, create an equation for the company's daily profit.

16. Compare the three forms of the equation of a quadratic relation using this concept circle. Under what conditions would you use one form instead of the other forms when trying to connect a graph to its equation? Explain your thinking.



Extending

17. The following transformations are applied to a parabola with the equation $y = 2(x + 3)^2 - 1$. Determine the equation that will result after each transformation.

a) a translation 4 units right
b) a reflection in the x-axis
c) a reflection in the x-axis, followed by a translation 5 units down
d) a stretch by a factor of 6
e) a compression by a factor of $\frac{1}{4}$, followed by a reflection in the y-axis

18. The vertex of the parabola $y = 3x^2 + bx + c$ is at $(-1, 4)$. Determine the values of b and c.

19. Determine an algebraic expression for the solution, x, to the equation $0 = a(x - h)^2 + k$. Do not expand the equation.