4.3 Factoring Quadratics: $x^2 + bx + c$

GOAL

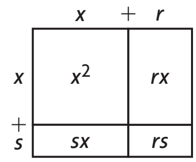
Factor quadratic expressions of the form $ax^2 + bx + c$, where $a = 1$.

YOU WILL NEED

- algebra tiles

INVESTIGATE the Math

Brigitte remembered that an area model can be used to multiply two binomials. To multiply $(x + r)(x + s)$, she created the model at the right and determined that the product is quadratic.



? How can an area model be used to determine the factors of a quadratic expression?

A. Use algebra tiles to build rectangles with the dimensions shown in the table below. Copy and complete the table, recording the area in the form $x^2 + bx + c$.

Length	Width	Area: $x^2 + bx + c$	Value of b	Value of c
$x + 3$	$x + 4$			
$x + 3$	$x + 5$			
$x + 3$	$x + 6$			
$x + 4$	$x + 4$			
$x + 4$	$x + 5$			

B. Look for a pattern in the table for part A. Use this pattern to predict the length and width of a rectangle with each of the following areas.

i) $x^2 + 8x + 12$
ii) $x^2 + 10x + 21$
iii) $x^2 + 11x + 30$
iv) $x^2 + 11x + 18$

C. The length of a rectangle is $x + 4$, and the width is $x - 3$. What is the area of the rectangle?

D. Repeat part A for rectangles with the following dimensions.

Length	Width	Area: $x^2 + bx + c$	Value of b	Value of c
$x - 3$	$x + 5$			
$x + 3$	$x - 6$			
$x - 2$	$x - 2$			
$x - 1$	$x - 5$			

E. Look for a pattern in the table for part D. Use this pattern to predict the length and width of a rectangle with each of the following areas.

i) $x^2 - 2x - 15$	**iii)** $x^2 - x - 30$
ii) $x^2 + 2x - 24$	**iv)** $x^2 - 8x + 7$

F. The expression $x^2 + bx + c$ represents the area of a rectangle. How can you factor this expression to predict the length and width of the rectangle?

Reflecting

G. Examine the length, width, and area of each rectangle in parts A and D. Explain how the signs in the area expression can be used to determine the signs in each dimension.

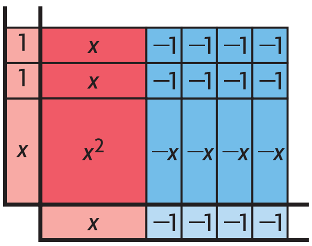
H. Can all quadratic expressions of the form $x^2 + bx + c$ be factored as the product of two binomials? Explain.

APPLY the Math

EXAMPLE 1 Selecting an algebra tile strategy to factor a quadratic expression

Factor $x^2 - 2x - 8$.

Timo's Solution



$x^2 - 2x - 8 = (x - 4)(x + 2)$

I arranged one x^2 tile, four $-x$ tiles, two x tiles, and eight negative unit tiles in a rectangle to create an area model.

I placed the x tiles and unit tiles so the length and width were easier to see. The dimensions of the rectangle are $x - 4$ and $x + 2$.

The sum $2 + (-4) = -2$ determines the number of x tiles, and the product $2 \times (-4) = -8$ determines the number of unit tiles in the original expression.

EXAMPLE 2 Connecting the factors of a trinomial to its coefficients and constants

Factor $x^2 + 12x + 35$.

Chaniqua's Solution

$x^2 + 12x + 35$
$= (x\,?\,)(x\,?\,)$

The two factors of the quadratic expression must be binomials that start with x. I need two numbers whose sum is 12 (the coefficient of x) and whose product is 35 (the constant).

$= (x + ?)(x + ?)$

I started with the product. Since 35 is positive, both numbers must be either positive or negative.

Since the sum is positive, both numbers must be positive.

$= (x + 7)(x + 5)$

The numbers are 7 and 5.

Check:

$(x + 7)(x + 5) = x^2 + 5x + 7x + 35$

$= x^2 + 12x + 35$

I checked by multiplying.

EXAMPLE 3 Reasoning to factor quadratic expressions

Factor each expression, if possible.

a) $x^2 - x - 72$ **b)** $a^2 - 13a + 36$ **c)** $x^2 + x + 6$

Ryan's Solution

a) $x^2 - x - 72$

$= (x - 9)(x + 8)$

I needed two numbers whose sum is −1 and whose product is −72.

The product is negative, so one of the numbers must be negative.

Since the sum is negative, the negative number must be farther from zero than the positive number.

The numbers are −9 and 8.

b) $a^2 - 13a + 36$

$= (a - 9)(a - 4)$

I needed two numbers whose sum is −13 and whose product is 36.

The product is positive, so both numbers must be either positive or negative.

Since the sum is negative, both numbers must be negative.

The numbers are −9 and −4.

c) $x^2 + x + 6$

This cannot be factored.

I needed two numbers whose sum is 1 and whose product is 6.

There are no such numbers because the only factors of 6 are 1 and 6, and 2 and 3. The sum of 1 and 6 is 7, and the sum of 2 and 3 is 5. Neither sum is 1.

EXAMPLE 4 Reasoning to factor a quadratic that has a common factor

Factor $3y^3 - 21y^2 - 24y$.

Sook Lee's Solution

$3y^3 - 21y^2 - 24y$

First, I divided out the greatest common factor. The GCF is $3y$, since all the terms are divisible by $3y$.

$= 3y(y^2 - 7y - 8)$

$= 3y(y - 8)(y + 1)$

To factor the trinomial, I needed two numbers whose sum is -7 and whose product is -8. These numbers are -8 and 1.

In Summary

Key Idea

- If a quadratic expression of the form $x^2 + bx + c$ can be factored, it can be factored into two binomials, $(x + r)$ and $(x + s)$, where $r + s = b$ and $r \times s = c$, and r and s are integers.

Need to Know

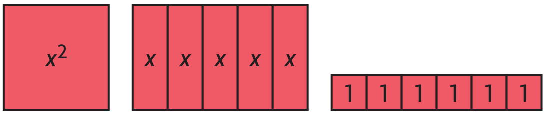
- To factor $x^2 + bx + c$ as $(x + r)(x + s)$, you can use the signs in the trinomial to determine the signs in the factors.

Trinomial	Factors
b and c are positive.	$(x + r)(x + s)$
b is negative, and c is positive.	$(x - r)(x - s)$
b and c are negative.	$(x - r)(x + s)$, where $r > s$
b is positive, and c is negative.	$(x + r)(x - s)$, where $r > s$

- It is easier to factor an algebraic expression if you first divide out the greatest common factor.

CHECK Your Understanding

1. a) Write the trinomial that is represented by these algebra tiles.

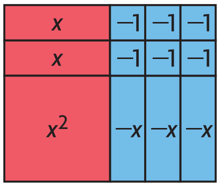


b) Sketch what the tiles would look like if they were arranged in a rectangle.

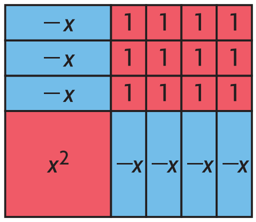
c) Use your sketch to determine the factors of the trinomial.

2. The tiles in each model represent an algebraic expression. Identify the expression and its factors.

a)



b)



3. One factor is given, and one factor is missing. What is the missing factor?

a) $x^2 - 10x + 21 = (x - 7)(\blacksquare)$

b) $x^2 + 4x - 32 = (x - 4)(\blacksquare)$

c) $x^2 - 2x - 63 = (\blacksquare)(x + 7)$

d) $x^2 + 14x + 45 = (\blacksquare)(x + 9)$

PRACTISING

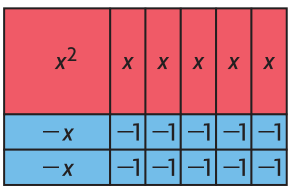
4. Factor each expression.

a) $x^2 + 2x + 1$ **c)** $x^2 - 2x - 3$ **e)** $x^2 - 4x + 4$

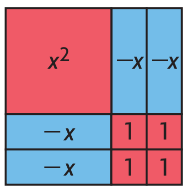
b) $x^2 - 2x + 1$ **d)** $x^2 + 6x + 9$ **f)** $x^2 - 4x - 12$

5. The tiles in each model represent a quadratic expression. Identify the expression and its factors.

a)



b)



6. One factor is given, and one factor is missing. What is the missing factor?

a) $x^2 + 11x + 24 = (x + 3)(\blacksquare)$
b) $c^2 - 15c + 56 = (c - 7)(\blacksquare)$
c) $a^2 - 11a - 60 = (a - 15)(\blacksquare)$
d) $y^2 - 20y - 44 = (\blacksquare)(y + 2)$
e) $b^2 + 2b - 48 = (\blacksquare)(b + 8)$
f) $z^2 - 19z + 90 = (\blacksquare)(z - 10)$

7. Factor each expression.

a) $x^2 + 4x + 3$
b) $a^2 - 9a + 20$
c) $m^2 - 8m + 16$
d) $n^2 + n - 6$
e) $x^2 + 6x - 16$
f) $x^2 + 15x - 16$

8. Factor.

a) $x^2 - 10x + 16$
b) $y^2 + 6y - 40$
c) $a^2 - a - 56$
d) $w^2 - 5w - 14$
e) $m^2 - 12m + 32$
f) $n^2 + n - 42$

9. Factor.

a) $3x^2 + 24x + 45$
b) $2y^2 - 2y - 60$
c) $3v^2 + 9v + 6$
d) $6n^2 + 24n - 30$
e) $x^3 + 5x^2 + 4x$
f) $7x^4 + 28x^3 - 147x^2$

10. K Write three different quadratic trinomials that have $(x - 2)$ as a factor.

11. C Nathan factored $x^2 - 15x + 44$ as $(x - 4)(x - 11)$. Martina factored the expression another way and found different factors. Identify the factors that Martina found, and explain why both students are correct.

12. Factor.

a) $a^2 + 8a + 15$
b) $3x^2 - 21x - 54$
c) $z^2 - 16z + 55$
d) $x^2 + 5x - 50$
e) $x^3 - 3x^2 - 10x$
f) $2xy^2 - 26xy + 84x$

13. Examine each quadratic relation below.

i) Express the relation in factored form.
ii) Determine the zeros.
iii) Determine the coordinates of the vertex.
iv) Sketch the graph of the relation.

a) $y = x^2 + 2x - 8$
b) $y = x^2 - 2x - 24$
c) $y = x^2 - 8x + 15$
d) $y = -x^2 - 9x - 14$

14. A professional cliff diver's height above the water can be modelled by the equation $h = -5t^2 + 20$, where h is the diver's height in metres and t is the elapsed time in seconds.
A

a) Draw a height versus time graph.
b) Determine the height of the cliff that the diver jumped from.
c) Determine when the diver will enter the water.

15. A baseball is thrown from the top of a building and falls to the ground below. The height of the baseball above the ground is approximated by the relation $h = -5t^2 + 10t + 40$, where h is the height above the ground in metres and t is the elapsed time in seconds. Determine the maximum height that is reached by the ball.

16. Factor each expression.
a) $m^2 + 4mn - 5n^2$
b) $x^2 + 12xy + 35y^2$
c) $a^2 + ab - 12b^2$
d) $c^2 - 12cd - 85d^2$
e) $r^2 + 13rs + 12s^2$
f) $18p^2 - 9pq + q^2$

17. Paul says that if you can factor $x^2 - bx - c$, you can factor $x^2 + bx - c$. Do you agree? Explain.
T

18. Create a mind map that shows the connections between $x^2 + bx + c$ and its factors.

Extending

19. Factor each expression.
a) $x^4 + 6x^2 - 27$
b) $a^4 + 10a^2 + 9$
c) $-4m^4 + 16m^2n^2 + 20n^4$
d) $(a - b)^2 - 15(a - b) + 26$

20. Factor, and then simplify. Assume that the denominator is never zero.
a) $\dfrac{x^2 - 6x + 8}{x - 4}$
b) $\dfrac{a^2 - 3a - 28}{a + 4}$
c) $\dfrac{x^2 + x - 30}{x - 5}$
d) $\dfrac{2x^2 - 24x + 64}{2x - 16}$