Chapter 3

Algebraic expansion and simplification

Contents:

A Collecting like terms
B Product notation
C The distributive law
D The product $(a+b)(c+d)$
E Difference of two squares
F Perfect squares expansion
G Further expansion
H The binomial expansion

Example 15 Self Tutor

Expand and simplify: **a** $(2x^2+3)^2$ **b** $5-(x+2)^2$

a
$$\begin{aligned}&(2x^2+3)^2\\&=(2x^2)^2+2\times 2x^2\times 3+3^2\\&=4x^4+12x^2+9\end{aligned}$$

b
$$\begin{aligned}&5-(x+2)^2\\&=5-[x^2+4x+4]\\&=5-x^2-4x-4\\&=1-x^2-4x\end{aligned}$$

Notice the use of square brackets in the second line. These remind us to change the signs inside them when they are removed.

5 Expand and simplify:

a $(x^2+2)^2$ **b** $(y^2-3)^2$ **c** $(3a^2+4)^2$

d $(1-2x^2)^2$ **e** $(x^2+y^2)^2$ **f** $(x^2-a^2)^2$

6 Expand and simplify:

a $3x+1-(x+3)^2$ **b** $5x-2+(x-2)^2$

c $(x+2)(x-2)+(x+3)^2$ **d** $(x+2)(x-2)-(x+3)^2$

e $(3-2x)^2-(x-1)(x+2)$ **f** $(1-3x)^2+(x+2)(x-3)$

g $(2x+3)(2x-3)-(x+1)^2$ **h** $(4x+3)(x-2)-(2-x)^2$

i $(1-x)^2+(x+2)^2$ **j** $(1-x)^2-(x+2)^2$

G FURTHER EXPANSION

In this section we expand more complicated expressions by repeated use of the expansion laws.

Consider the expansion of $(a+b)(c+d+e)$.

Now
$$\begin{aligned}&(a+b)(c+d+e)\\&=(a+b)c+(a+b)d+(a+b)e\\&=ac+bc+ad+bd+ae+be\end{aligned}$$

Compare:
$$\begin{aligned}&\square(c+d+e)\\&=\square c+\square d+\square e\end{aligned}$$

Notice that there are 6 terms in this expansion and that each term within the first bracket is multiplied by each term in the second.

2 terms in the first bracket $\times$ 3 terms in the second bracket $\longrightarrow$ 6 terms in the expansion.

Example 16 Self Tutor

Expand and simplify: $(2x+3)(x^2+4x+5)$

$$(2x+3)(x^2+4x+5)$$

$$\begin{aligned}&=2x^3+8x^2+10x && \{\text{all terms of 2nd bracket}\times 2x\}\\&\quad+3x^2+12x+15 && \{\text{all terms of 2nd bracket}\times 3\}\\&=2x^3+11x^2+22x+15 && \{\text{collecting like terms}\}\end{aligned}$$

Example 17 Self Tutor

Expand and simplify: $(x+2)^3$

$$\begin{aligned}(x+2)^3 &= (x+2) \times (x+2)^2 \\ &= (x+2)(x^2+4x+4) \\ &= x^3+4x^2+4x && \{\text{all terms in 2nd bracket} \times x\} \\ &\quad +2x^2+8x+8 && \{\text{all terms in 2nd bracket} \times 2\} \\ &= x^3+6x^2+12x+8 && \{\text{collecting like terms}\}\end{aligned}$$

Example 18 Self Tutor

Expand and simplify:

a $x(x+1)(x+2)$ **b** $(x+1)(x-2)(x+2)$

a
$$\begin{aligned}&x(x+1)(x+2) \\ &= (x^2+x)(x+2) && \{\text{all terms in first bracket} \times x\} \\ &= x^3+2x^2+x^2+2x && \{\text{expanding remaining factors}\} \\ &= x^3+3x^2+2x && \{\text{collecting like terms}\}\end{aligned}$$

b
$$\begin{aligned}&(x+1)(x-2)(x+2) \\ &= (x+1)(x^2-4) && \{\text{difference of two squares}\} \\ &= x^3-4x+x^2-4 && \{\text{expanding factors}\} \\ &= x^3+x^2-4x-4\end{aligned}$$

Always look for ways to make your expansions simpler. In **b** we can use the difference of two squares.

EXERCISE 3G

1 Expand and simplify:

a $(x+3)(x^2+x+2)$ **b** $(x+4)(x^2+x-2)$

c $(x+2)(x^2+x+1)$ **d** $(x+5)(x^2-x-1)$

e $(2x+1)(x^2+x+4)$ **f** $(3x-2)(x^2-x-3)$

g $(x+2)(2x^2-x+2)$ **h** $(2x-1)(3x^2-x+2)$

Each term of the first bracket is multiplied by each term of the second bracket.

2 Expand and simplify:

a $(x+1)^3$ **b** $(x+3)^3$ **c** $(x-1)^3$

d $(x-3)^3$ **e** $(2x+1)^3$ **f** $(3x-2)^3$

3 Expand and simplify:

a $x(x+2)(x+3)$ **b** $x(x-4)(x+1)$ **c** $x(x-3)(x-2)$

d $2x(x+3)(x+1)$ **e** $2x(x-4)(1-x)$ **f** $-x(3+x)(2-x)$

g $-3x(2x-1)(x+2)$ **h** $x(1-3x)(2x+1)$ **i** $2x^2(x-1)^2$

4 Expand and simplify:

a $(x+3)(x+2)(x+1)$ **b** $(x-2)(x-1)(x+4)$

c $(x-4)(x-1)(x-3)$ **d** $(2x-1)(x+2)(x-1)$

e $(3x+2)(x+1)(x+3)$ **f** $(2x+1)(2x-1)(x+4)$

g $(1-x)(3x+2)(x-2)$ **h** $(x-3)(1-x)(3x+2)$

H THE BINOMIAL EXPANSION

Consider $(a+b)^n$. We note that:

- $a+b$ is called a **binomial** as it contains two terms
- any expression of the form $(a+b)^n$ is called a **power of a binomial**
- the **binomial expansion** of $(a+b)^n$ is obtained by writing the expression without brackets.

Now
$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

So, the **binomial expansion** of $\boldsymbol{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$.

Example 19 Self Tutor

Expand and simplify using the rule $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$:

a $(x+2)^3$ **b** $(2x-1)^3$

a We substitute $a=x$ and $b=2$

$$\begin{aligned}\therefore\ (x+2)^3 &= x^3+3\times x^2\times 2+3\times x\times 2^2+2^3\\ &= x^3+6x^2+12x+8\end{aligned}$$

b We substitute $a=(2x)$ and $b=(-1)$

$$\begin{aligned}\therefore\ (2x-1)^3 &= (2x)^3+3\times(2x)^2\times(-1)+3\times(2x)\times(-1)^2+(-1)^3\\ &= 8x^3-12x^2+6x-1\end{aligned}$$

EXERCISE 3H

1 Use the binomial expansion for $(a+b)^3$ to expand and simplify:

a $(x+1)^3$ **b** $(a+3)^3$ **c** $(x+5)^3$

d $(x-1)^3$ **e** $(x-2)^3$ **f** $(x-3)^3$

g $(3+a)^3$ **h** $(3x+2)^3$ **i** $(2x+3y)^3$

2 Copy and complete the argument $(a+b)^4 = (a+b)(a+b)^3$
$= (a+b)(a^3+3a^2b+3ab^2+b^3)$
$\vdots$

3 Use the binomial expansion $(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$ to expand and simplify:

a $(x+1)^4$ **b** $(y+2)^4$ **c** $(3+a)^4$ **d** $(b+4)^4$
e $(x-1)^4$ **f** $(y-2)^4$ **g** $(3-a)^4$ **h** $(b-4)^4$

4 Find the binomial expansion of $(a+b)^5$ by considering $(a+b)(a+b)^4$. Hence, write down the binomial expansion for $(a-b)^5$.

REVIEW SET 3A

1 Expand and simplify:

a $4x \times -8$ **b** $5x \times 2x^2$ **c** $-4x \times -6x$
d $3x \times x - 2x^2$ **e** $4a \times c + 3c \times a$ **f** $2x^2 \times x - 3x \times x^2$

2 Expand and simplify:

a $-3(x+6)$ **b** $2x(x^2-4)$
c $2(x-5)+3(2-x)$ **d** $3(1-2x)-(x-4)$
e $2x-3x(x-2)$ **f** $x(2x+1)-2x(1-x)$
g $x^2(x+1)-x(1-x^2)$ **h** $9(a+b)-a(4-b)$

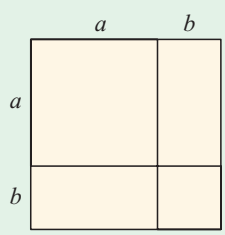
3 Expand and simplify:

a $(3x+2)(x-2)$ **b** $(2x-1)^2$ **c** $(4x+1)(4x-1)$
d $(5-x)^2$ **e** $(3x-7)(2x-5)$ **f** $x(x+2)(x-2)$
g $(3x+5)^2$ **h** $-(x-2)^2$ **i** $-2x(x-1)^2$

4 Expand and simplify:

a $5+2x-(x+3)^2$ **b** $(x+2)^3$
c $(3x-2)(x^2+2x+7)$ **d** $(x-1)(x-2)(x-3)$
e $x(x+1)^3$ **f** $(x^2+1)(x-1)(x+1)$

5



Explain how to use the given figure to show that $(a+b)^2 = a^2+2ab+b^2$.