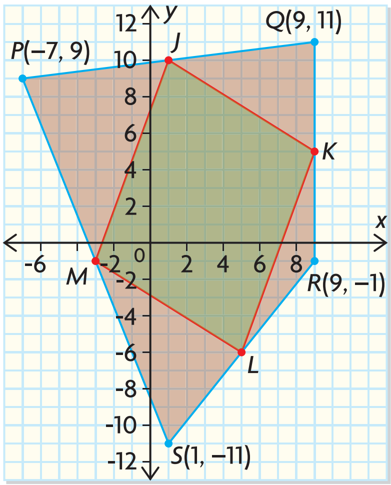
2.5 Verifying Properties of Geometric Figures

YOU WILL NEED

- grid paper and ruler, or dynamic geometry software



GOAL

Use analytic geometry to verify properties of geometric figures.

LEARN ABOUT the Math

Carlos has hired a landscape designer to give him some ideas for improving his backyard, which is a quadrilateral. The designer's plan on a coordinate grid shows a lawn area that is formed by joining the midpoints of the adjacent sides in the quadrilateral. The four triangular areas will be gardens.

? How can Carlos verify that the lawn area is a parallelogram?

EXAMPLE 1 **Proving** a conjecture about a geometric figure

Show that the **midsegments of the quadrilateral**, with vertices at $P(-7, 9)$, $Q(9, 11)$, $R(9, -1)$, and $S(1, -11)$, form a parallelogram.

midsegment of a quadrilateral
a line segment that connects the midpoints of two adjacent sides in a quadrilateral

Ed's Solution: Using slopes

J has coordinates $\left(\dfrac{-7 + 9}{2}, \dfrac{9 + 11}{2}\right) = (1, 10).$

K has coordinates $\left(\dfrac{9 + 9}{2}, \dfrac{11 + (-1)}{2}\right) = (9, 5).$

L has coordinates $\left(\dfrac{9 + 1}{2}, \dfrac{-1 + (-11)}{2}\right) = (5, -6).$

M has coordinates $\left(\dfrac{1 + (-7)}{2}, \dfrac{-11 + 9}{2}\right) = (-3, -1).$

I used the midpoint formula to determine the coordinates of the midpoints of *PQ*, *QR*, *RS*, and *SP*, which are *J*, *K*, *L*, and *M*.

$$m_{JK} = \frac{5 - 10}{9 - 1} = -0.625 \qquad m_{LM} = \frac{-1 - (-6)}{-3 - 5} = -0.625$$

$$m_{KL} = \frac{-6 - 5}{5 - 9} = 2.75 \qquad m_{MJ} = \frac{10 - (-1)}{1 - -3} = 2.75$$

I needed to show that *JK* is parallel to *LM* and that *KL* is parallel to *MJ*.

I used the slope formula, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, to calculate the slopes of *JK*, *KL*, *LM*, and *MJ*.

$m_{JK} = m_{LM}$ and $m_{KL} = m_{MJ}$
$JK \parallel LM$ and $KL \parallel MJ$

Quadrilateral *JKLM* is a parallelogram.

I saw that the slopes of *JK* and *LM* are the same and the slopes of *KL* and *MJ* are the same. This means that the opposite sides in quadrilateral *JKLM* are parallel. So quadrilateral *JKLM* must be a parallelogram.

Grace's Solution: Using properties of the diagonals

J has coordinates $\left(\frac{-7 + 9}{2}, \frac{9 + 11}{2}\right) = (1, 10)$.

K has coordinates $\left(\frac{9 + 9}{2}, \frac{11 + (-1)}{2}\right) = (9, 5)$.

L has coordinates $\left(\frac{9 + 1}{2}, \frac{-1 + (-11)}{2}\right) = (5, -6)$.

M has coordinates $\left(\frac{1 + (-7)}{2}, \frac{-11 + 9}{2}\right) = (-3, -1)$.

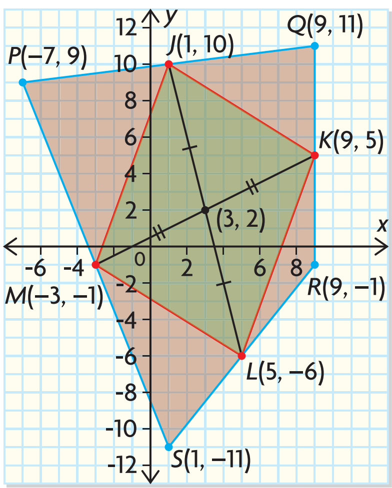
I calculated the coordinates of points *J*, *K*, *L*, and *M*, the midpoints of the sides in quadrilateral *PQRS*.

The midpoint of JL is $\left(\frac{1 + 5}{2}, \frac{10 + (-6)}{2}\right) = (3, 2)$.

The midpoint of KM is $\left(\frac{9 + (-3)}{2}, \frac{5 + (-1)}{2}\right) = (3, 2)$.

Then I calculated the midpoints of the diagonals *JL* and *KM*.

I discovered that both diagonals have the same midpoint, so they must intersect at this point.



The diagonals of the quadrilateral bisect each other since they have the same midpoint.

This means that *JKLM* must be a parallelogram.

JKLM is a parallelogram.

Reflecting

A. How is Ed's strategy different from Grace's strategy?

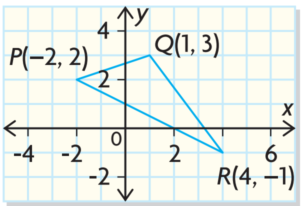
B. What is another strategy you could use to show that *JKLM* is a parallelogram?

APPLY the Math

EXAMPLE 2 **Selecting a strategy** to verify a property of a triangle

A triangle has vertices at $P(-2, 2)$, $Q(1, 3)$, and $R(4, -1)$. Show that the midsegment joining the midpoints of PQ and PR is parallel to QR and half its length.

Andrea's Solution: Using slopes and lengths of line segments



I drew a diagram of the triangle.

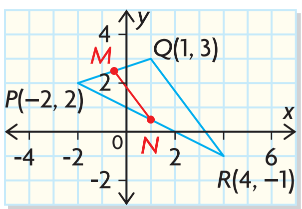
The midpoint of PQ is

$$M\left(\frac{-2 + 1}{2}, \frac{2 + 3}{2}\right) = M(-0.5, 2.5).$$

The midpoint of PR is

$$N\left(\frac{-2 + 4}{2}, \frac{2 + (-1)}{2}\right) = N(1, 0.5).$$

I determined the midpoints of PQ and PR. I used M for the midpoint of PQ and N for the midpoint of PR.



I drew the line segment that joins M to N.

I knew that the slopes of QR and MN would be the same if QR is parallel to MN.

$$m_{QR} = \frac{-1 - 3}{4 - 1}$$
$$= -\frac{4}{3}$$
$$m_{MN} = \frac{0.5 - 2.5}{1 - (-0.5)}$$
$$= \frac{-2}{1.5}$$
$$= -\frac{4}{3}$$

$MN \parallel QR$

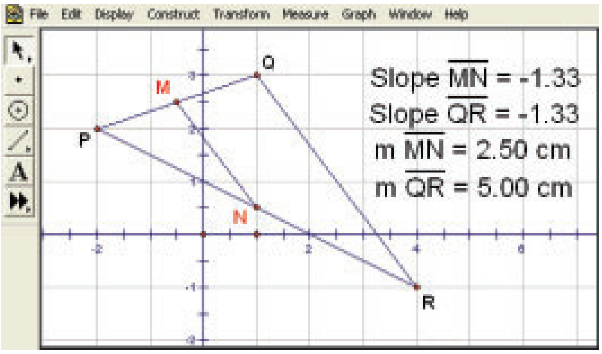
I calculated the slopes of QR and MN.

I multiplied $\frac{-2}{1.5}$ by $\frac{2}{2}$ to get $-\frac{4}{3}$. The slopes are the same, so MN is parallel to QR.

$$\begin{aligned} QR &= \sqrt{(4-1)^2 + (-1-3)^2} \\ &= \sqrt{9+16} \\ &= \sqrt{25} \\ &= 5 \\ MN &= \sqrt{[1-(-0.5)]^2 + (0.5-2.5)^2} \\ &= \sqrt{2.25+4} \\ &= \sqrt{6.25} \\ &= 2.5 \\ MN &= \frac{1}{2}QR \end{aligned}$$

Next, I calculated the lengths of *QR* and *MN*. The length of *MN* is exactly one-half the length of *QR*.

The midsegment that joins the midpoints of *PQ* and *PR* is parallel to *QR* and one-half its length.



I verified my calculations using dynamic geometry software. I chose a scale where 1 unit = 1 cm. I constructed the triangle and the midsegment *MN*. Then I measured the lengths and slopes of *MN* and *QR*. My calculations were correct.

EXAMPLE 3 Reasoning about lines and line segments to verify a property of a circle

Show that points $A(10, 5)$ and $B(2, -11)$ lie on the circle with equation $x^2 + y^2 = 125$. Also show that the perpendicular bisector of **chord** *AB* passes through the centre of the circle.

Drew's Solution

$r = \sqrt{125}$

$r \doteq 11.2$

The intercepts are located at (0, 11.2), (0, −11.2), (11.2, 0), and (−11.2, 0).

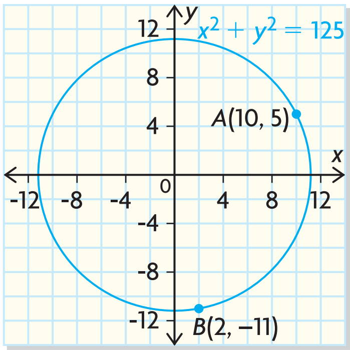
I knew that $x^2 + y^2 = 125$ is the equation of a circle with centre (0, 0) since it is in the form $x^2 + y^2 = r^2$.

I calculated the radius and used this value to determine the coordinates of the intercepts.

Left Side	Right Side	Left Side	Right Side
$x^2 + y^2$	125	$x^2 + y^2$	125
$= 10^2 + 5^2$		$= 2^2 + (-11)^2$	
$= 125$		$= 125$	

I substituted the coordinates of points *A* and *B* into the equation of the circle to show that *A* and *B* are on the circle.

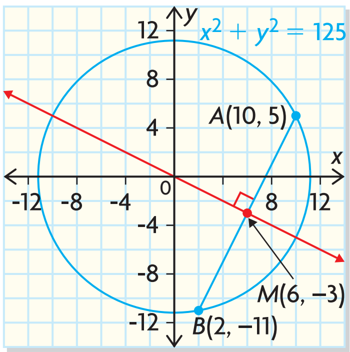
Points $A(10, 5)$ and $B(2, -11)$ lie on the circle.



I used the intercepts to sketch the circle. I marked points A and B on the circle.

The midpoint of AB is

$$M\left(\frac{10 + 2}{2}, \frac{5 + (-11)}{2}\right) = M(6, -3).$$



I determined the midpoint and marked it on my sketch. I called the midpoint M. Then I sketched the perpendicular bisector of AB.

To write an equation for the perpendicular bisector, I had to know its slope and the coordinates of a point on it. I already knew that the midpoint $M(6, -3)$ is on the perpendicular bisector.

$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-11 - 5}{2 - 10} \\ &= \frac{-16}{-8} \end{aligned}$$

To determine the slope of the perpendicular bisector, I had to calculate the slope of AB.

I knew that the slope of the perpendicular bisector is the negative reciprocal of 2, which is $-\frac{1}{2}$.

The slope of chord AB is 2.

The slope of the perpendicular bisector is $-\frac{1}{2}$.

An equation for the perpendicular bisector is

$y = -\frac{1}{2}x + b.$

Since $M(6, -3)$ lies on the perpendicular bisector,

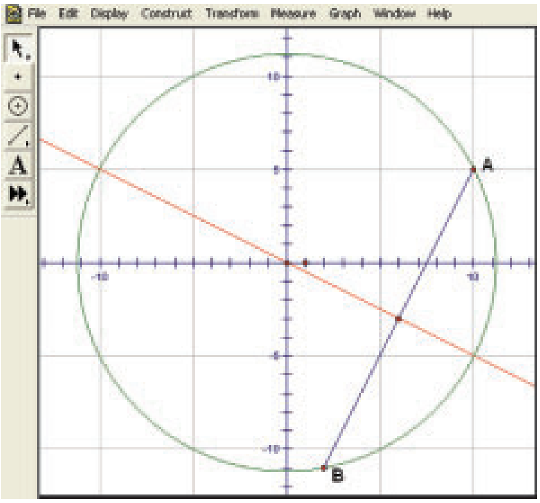
$$\begin{aligned} -3 &= -\frac{1}{2}(6) + b \\ -3 &= -3 + b \\ 0 &= b \end{aligned}$$

I wrote the equation in the form $y = mx + b$. Then I substituted the coordinates of M into the equation to determine the value of b.

Since $b = 0$, the line goes through (0, 0), which is the centre of the circle.

The equation of the perpendicular bisector of chord AB is $y = -\frac{1}{2}x$. The y-intercept is 0.

The line passes through (0, 0), which is the centre of the circle.



I verified my calculations using dynamic geometry software. I constructed the circle, the chord, and the perpendicular bisector of the chord. The sketch confirmed that the perpendicular bisector passes through the centre of the circle.

In Summary

Key Idea

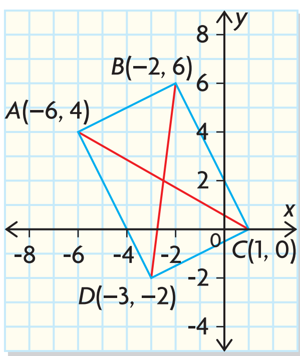
- When you draw a geometric figure on a coordinate grid, you can verify many of its properties using the properties of lines and line segments.

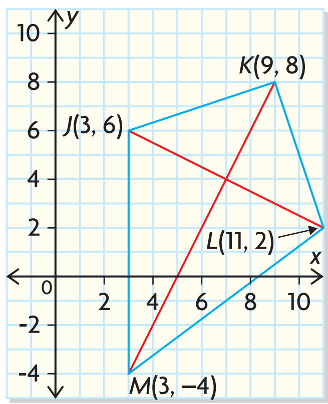
Need to Know

- You can use the midpoint formula to determine whether a point bisects a line segment.
- You can use the formula for the length of a line segment to calculate the lengths of two or more sides in a geometric figure so that you can compare them.
- You can use the slope formula to determine whether the sides in a geometric figure are parallel, perpendicular, or neither.

CHECK Your Understanding

1. Show that the diagonals of quadrilateral *ABCD* at the right are equal in length.
2. Show that the diagonals of quadrilateral *JKLM* at the far right are perpendicular.
3. $\triangle PQR$ has vertices at $P(-2, 1)$, $Q(1, 5)$, and $R(5, 2)$. Show that the median from vertex Q is the perpendicular bisector of PR.





PRACTISING

4. A rectangle has vertices at $J(10, 0)$, $K(-8, 6)$, $L(-12, -6)$, and $M(6, -12)$. Show that the diagonals bisect each other.

5. A rectangle has vertices at $A(-6, 5)$, $B(12, -1)$, $C(8, -13)$, and
K $D(-10, -7)$. Show that the diagonals are the same length.

6. Make a conjecture about the type of quadrilateral shown in question 1. Use
C analytic geometry to explain why your conjecture is either true or false.

7. Make a conjecture about the type of quadrilateral shown in question 2. Use analytic geometry to explain why your conjecture is either true or false.

8. A triangle has vertices at $D(-5, 4)$, $E(1, 8)$, and $F(-1, -2)$. Show that the height from D is also the median from D.

9. Show that the midsegments of a quadrilateral with vertices at $P(-2, -2)$, $Q(0, 4)$, $R(6, 3)$, and $S(8, -1)$ form a rhombus.

10. Show that the midsegments of a rhombus with vertices at $R(-5, 2)$, $S(-1, 3)$, $T(-2, -1)$, and $U(-6, -2)$ form a rectangle.

11. Show that the diagonals of the rhombus in question 10 are perpendicular and bisect each other.

12. Show that the midsegments of a square with vertices at $A(2, -12)$, $B(-10, -8)$, $C(-6, 4)$, and $D(6, 0)$ form a square.

13. **a)** Show that points $A(-4, 3)$ and $B(3, -4)$ lie on $x^2 + y^2 = 25$.
b) Show that the perpendicular bisector of chord AB passes through the centre of the circle.

14. A trapezoid has vertices at $A(1, 2)$, $B(-2, 1)$, $C(-4, -2)$, and $D(2, 0)$.
A **a)** Show that the line segment joining the midpoints of BC and AD is parallel to both AB and DC.
b) Show that the length of this line segment is half the sum of the lengths of the parallel sides.

15. $\triangle ABC$ has vertices at $A(3, 4)$, $B(-2, 0)$, and $C(5, 0)$. Prove that the
T area of the triangle formed by joining the midpoints of $\triangle ABC$ is one-quarter the area of $\triangle ABC$.

16. Naomi claims that the midpoint of the hypotenuse of a right triangle is the same distance from each vertex of the triangle. Create a flow chart that summarizes the steps you would take to verify this property.

Extending

17. Show that the intersection of the line segments joining the midpoints of the opposite sides of a square is the same point as the midpoints of the diagonals.