2.4 Classifying Figures on a Coordinate Grid

YOU WILL NEED

- grid paper and ruler, or dynamic geometry software

GOAL

Use properties of line segments to classify two-dimensional figures.

LEARN ABOUT the Math

A surveyor has marked the corners of a lot where a building is going to be constructed. The corners have coordinates $P(-5,-5)$, $Q(-30, 10)$, $R(-5, 25)$, and $S(20, 10)$. Each unit represents 1 m. The builder wants to know the perimeter and shape of this building lot.

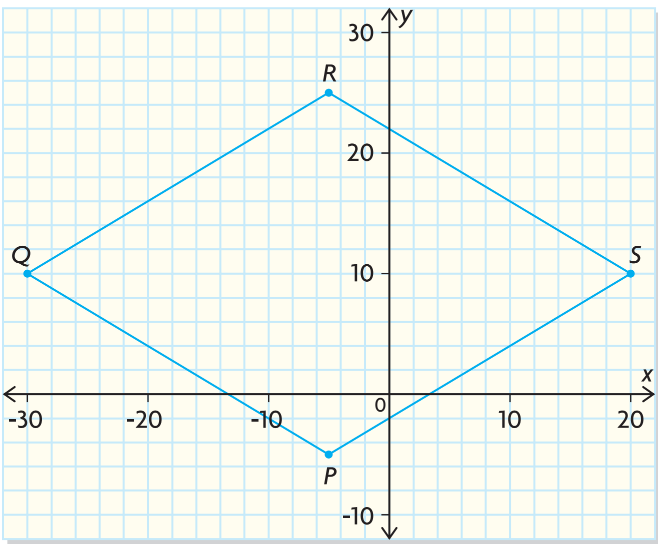
? How can the builder use the coordinates of the corners to determine the shape and perimeter of the lot?

EXAMPLE 1 **Connecting slopes and lengths of line segments to classifying a figure**

Use **analytic geometry** to identify the shape of quadrilateral *PQRS* and its perimeter.

analytic geometry
geometry that uses the *xy*-axes, algebra, and equations to describe relations and positions of geometric figures

Anita's Solution



I plotted the points on grid paper and then joined them to draw the figure.

I saw that the shape of the building lot looked like a parallelogram or a rhombus, but I couldn't be sure.

I knew that if *PQRS* was either of these figures, the opposite sides would be parallel. I also knew that if *PQRS* was a rhombus, all the sides would be the same length.

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

I decided to calculate the slope and the length of each side of *PQRS*.

$$\begin{aligned} PQ &= \sqrt{[-30 - (-5)]^2 + [10 - (-5)]^2} \\ &= \sqrt{625 + 225} \\ &= \sqrt{850} \\ &\doteq 29.15 \end{aligned}$$

The length of *PQ* is about 29.15 units.

$$\begin{aligned} m_{PQ} &= \frac{10 - (-5)}{-30 - (-5)} \\ &= \frac{15}{-25} \\ &= -\frac{3}{5} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{[-5 - (-30)]^2 + (25 - 10)^2} \\ &= \sqrt{625 + 225} \\ &= \sqrt{850} \\ &\doteq 29.15 \end{aligned}$$

The length of *QR* is about 29.15 units.

$$\begin{aligned} m_{QR} &= \frac{25 - 10}{-5 - (-30)} \\ &= \frac{15}{25} \\ &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} RS &= \sqrt{[20 - (-5)]^2 + (10 - 25)^2} \\ &= \sqrt{625 + 225} \\ &= \sqrt{850} \\ &\doteq 29.15 \end{aligned}$$

The length of *RS* is about 29.15 units.

$$\begin{aligned} m_{RS} &= \frac{10 - 25}{20 - (-5)} \\ &= \frac{-15}{25} \\ &= -\frac{3}{5} \end{aligned}$$

$$\begin{aligned} SP &= \sqrt{(-5 - 20)^2 + (-5 - 10)^2} \\ &= \sqrt{625 + 225} \\ &= \sqrt{850} \\ &\doteq 29.15 \end{aligned}$$

The length of *SP* is about 29.15 units.

$$\begin{aligned} m_{SP} &= \frac{-5 - 10}{-5 - 20} \\ &= \frac{-15}{-25} \\ &= \frac{3}{5} \end{aligned}$$

The slopes of *PQ* and *RS* are the same, so they are parallel. The slopes of *QR* and *SP* are also the same, so they are parallel too.

My length calculations showed that all four sides are equal.

$PQ \parallel RS$ and $QR \parallel SP$

$PQ = QR = RS = SP$

Since the opposite sides are parallel and all the side lengths are equal, *PQRS* is a rhombus.

$$\begin{aligned} \text{Perimeter} &\doteq 4(29.15) \\ &= 116.6 \end{aligned}$$

I multiplied the side length by 4 to calculate the perimeter.

The building lot is a rhombus. Its perimeter measures about 116.6 m.

Communication *Tip*

The symbol || is used to replace the phrase "is parallel to."

Reflecting

A. Why could Anita not rely completely on her diagram to determine the shape of the quadrilateral?

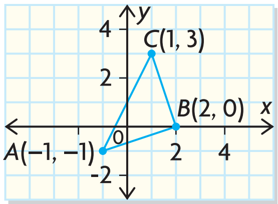
B. Why did Anita need to calculate the slopes and lengths of all the sides to determine the shape of the building lot? Explain.

APPLY the Math

EXAMPLE 2 **Reasoning about lengths and slopes to classify a triangle**

A triangle has vertices at $A(-1, -1)$, $B(2, 0)$, and $C(1, 3)$. What type of triangle is it? Justify your decision.

Angelica's Solution



I drew a diagram of the triangle on grid paper. I thought that the triangle might be isosceles since *AB* and *BC* look like they are the same length. The triangle might also be a right triangle since $\angle ABC$ looks like it might be 90°.

$$\begin{aligned} AB &= \sqrt{[2-(-1)]^2+[0-(-1)]^2} \\ &= \sqrt{3^2+1^2} \\ &= \sqrt{10} \\ &\doteq 3.2 \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(1-2)^2+(3-0)^2} \\ &= \sqrt{(-1)^2+3^2} \\ &= \sqrt{10} \\ &\doteq 3.2 \end{aligned}$$

To determine the type of triangle, I had to know the lengths of the sides. To check my isosceles prediction, I used the distance formula to determine the lengths of *AB* and *BC*. I rounded each answer to one decimal place.

AB and *BC* are the same length, so $\triangle ABC$ is isosceles.

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\begin{aligned} m_{AB} &= \frac{0-(-1)}{2-(-1)} \\ &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} m_{BC} &= \frac{3-0}{1-2} \\ &= \frac{3}{-1} \\ &= -3 \end{aligned}$$

To determine if the triangle has a right angle, I had to determine if a pair of sides are perpendicular. I did this by comparing their slopes. I calculated the slopes of the sides that looked perpendicular, *AB* and *BC*.

The slopes of *AB* and *BC* are negative reciprocals, so $AB \perp BC$. This means that $\triangle ABC$ is a right triangle.

$\triangle ABC$ is an isosceles right triangle, with $AB = BC$ and $\angle ABC = 90°$.

Communication *Tip*

The symbol $\perp$ is used to replace the phrase "is perpendicular to."

EXAMPLE 3 Solving a problem using properties of line segments

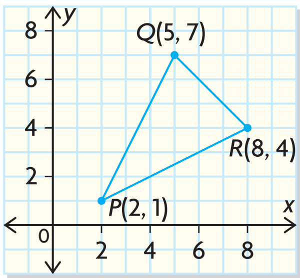
Tony is constructing a patterned concrete patio that is in the shape of an isosceles triangle, as requested by his client. On his plan, the vertices of the triangle are at $P(2, 1)$, $Q(5, 7)$, and $R(8, 4)$. Each unit represents 1 m.

a) Confirm that the plan shows an isosceles triangle.

b) Calculate the area of the patio.

Tony's Solution

a)



I made a sketch of the triangle. It looks isosceles since *PQ* and *RP* appear to be the same length.

$$\text{Length} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\begin{aligned} PQ &= \sqrt{(5 - 2)^2 + (7 - 1)^2} \\ &= \sqrt{45} \\ &\doteq 6.7 \end{aligned}$$

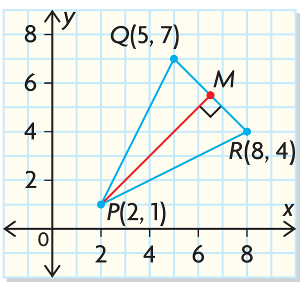
I used the distance formula to calculate the lengths of the sides of the triangle. I saw that *PQ* is the same length as *RP*, so the triangle is isosceles.

$$\begin{aligned} RP &= \sqrt{(2 - 8)^2 + (1 - 4)^2} \\ &= \sqrt{45} \\ &\doteq 6.7 \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(8 - 5)^2 + (4 - 7)^2} \\ &= \sqrt{18} \\ &\doteq 4.2 \end{aligned}$$

Since $PQ = RP$, the triangle is isosceles.

b)



I knew that I needed the lengths of the base and height to calculate the area of the triangle, since area $= \dfrac{\text{base} \times \text{height}}{2}$.

I remembered that, in an isosceles triangle, the median from the vertex where the two equal sides meet is perpendicular to the side that is opposite this vertex. *PM* is the height of the triangle and *QR* is its base.

Midpoint of QR is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{5 + 8}{2}, \frac{7 + 4}{2}\right)$$
$$= (6.5, 5.5)$$

I calculated M, the midpoint of QR, so I could use it to determine the length of PM.

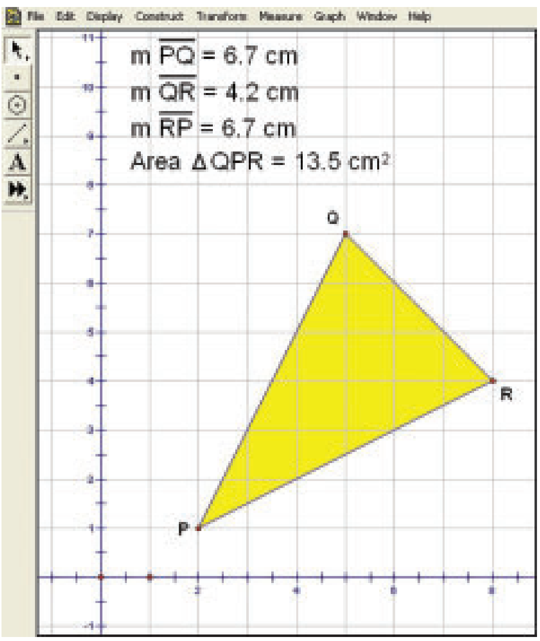
$$PM = \sqrt{(6.5 - 2)^2 + (5.5 - 1)^2}$$
$$= \sqrt{20.25 + 20.25}$$
$$= \sqrt{40.5}$$
$$\doteq 6.4$$

I used the distance formula to calculate the length of PM. I already knew that the length of QR is $\sqrt{18}$. PM is the height of the triangle, and QR is the base.

$$\text{Area of } \triangle PQR = \frac{QR \times PM}{2}$$
$$= \frac{\sqrt{18} \times \sqrt{40.5}}{2}$$
$$= 13.5$$

I calculated the area of the triangle using the exact values to minimize the rounding error.

The triangular patio has an area of 13.5 m^2.



I checked my calculations by plotting the vertices of the triangle using dynamic geometry software. Then I constructed the triangle and its interior.

I measured the lengths of the sides and determined the area. The scale in this sketch is 1 unit = 1 cm instead of 1 unit = 1 m. Since the numbers were the same, however, I knew that my calculations were correct.

Tech Support

Do not change the scale of the dynamic geometry software grid, because this will make the unit length different from 1 cm.

In Summary

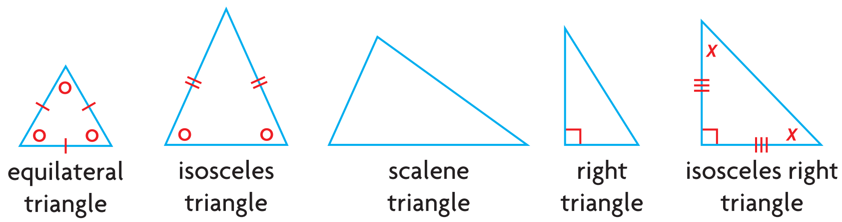
Key Idea

- When a geometric figure is drawn on a coordinate grid, the coordinates of its vertices can be used to calculate the slopes and lengths of the line segments, as well as the coordinates of the midpoints.

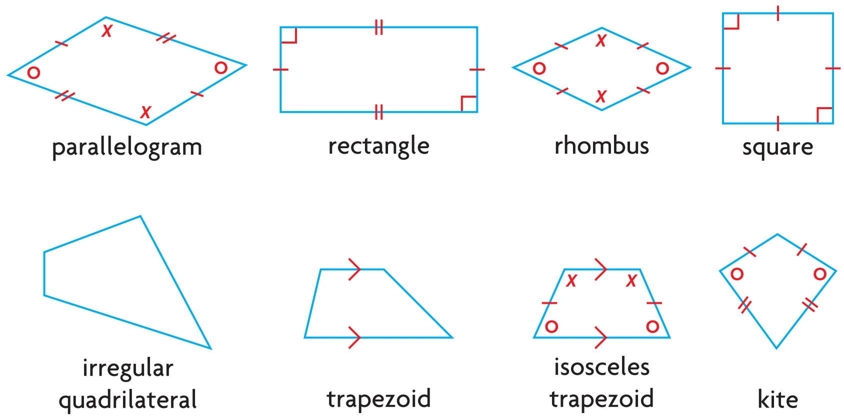
Need to Know

- Triangles and quadrilaterals can be classified by the relationships between their sides and their interior angles.

Triangles



Quadrilaterals



- To solve a problem that involves a geometric figure, it is a good idea to start by drawing a diagram of the situation on a coordinate grid.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are negative reciprocals.

CHECK Your Understanding

Round all answers to two decimal places, where necessary.

1. Show that the line segment joining points $P(1, 4)$ and $Q(5, 5)$ is parallel to the line segment joining points $R(3, -4)$ and $S(7, -3)$.

2. Show that TU, $T(-1, 7)$ and $U(3, 5)$, is perpendicular to VW, $V(-4, 1)$ and $W(-1, 7)$.

3. The sides of quadrilateral $ABCD$ have the following slopes.

Side	AB	BC	CD	AD
Slope	-5	$-\frac{1}{7}$	-5	$-\frac{1}{7}$

What types of quadrilateral could $ABCD$ be? What other information is needed to determine the exact type of quadrilateral?

4. $\triangle DEF$ has vertices at $D(-3, -4)$, $E(-2, 4)$, and $F(5, -5)$.

a) Show that $\triangle DEF$ is isosceles.
b) Determine the length of the median from vertex D.
c) Show that this median is perpendicular to EF.

PRACTISING

5. The lengths of the sides in a quadrilateral are $PQ = 4.5$ units, $QR = 4.5$ units, $RS = 4.5$ units, and $SP = 4.5$ units. What types of quadrilateral could $PQRS$ be? What other information is needed to determine the exact type of quadrilateral?

6. The following points are the vertices of triangles. Predict whether each triangle is scalene, isosceles, or equilateral. Then draw the triangle on a coordinate grid and calculate each side length to check your prediction.

a) $A(3, 3)$, $B(-1, 2)$, $C(0, -2)$
b) $G(-1, 3)$, $H(-2, -2)$, $I(2, 0)$
c) $D(2, -3)$, $E(-2, -4)$, $F(6, -6)$
d) $J(2, 5)$, $K(5, -2)$, $L(-1, -2)$

7. $P(-7, 1)$, $Q(-8, 4)$, and $R(-1, 3)$ are the vertices of a triangle. Show that $\triangle PQR$ is a right triangle.

8. A triangle has vertices at $L(-7, 0)$, $M(2, 1)$, and $N(-3, 5)$. Verify that it is a right isosceles triangle.

9. **a)** How can you use the distance formula to decide whether points $P(-2, -3)$, $Q(4, 1)$, and $R(2, 4)$ form a right triangle? Justify your answer.

b) Without drawing any diagrams, explain which sets of points are the vertices of right triangles.

i) $S(-2, 2)$, $T(-1, -2)$, $U(7, 0)$
ii) $X(3, 2)$, $Y(1, -2)$, $Z(-3, 6)$
iii) $A(5, 5)$, $B(3, 8)$, $C(8, 7)$

10. A quadrilateral has vertices at $W(-3, 2)$, $X(2, 4)$, $Y(6, -1)$, and $Z(1, -3)$.

a) Determine the length and slope of each side of the quadrilateral.
b) Based on your calculations for part a), what type of quadrilateral is $WXYZ$? Explain.
c) Determine the difference in the lengths of the two diagonals of $WXYZ$.

11. A polygon is defined by points $R(-5, 1)$, $S(5, 3)$, $T(2, -1)$, and $U(-8, -3)$. Show that the polygon is a parallelogram.

12. A quadrilateral has vertices at $A(-2, 3)$, $B(-2, -2)$, $C(2, 1)$, and $D(2, 6)$. Show that the quadrilateral is a rhombus.

13. **K** a) Show that $EFGH$, with vertices at $E(-2, 3)$, $F(2, 1)$, $G(0, -3)$, and $H(-4, -1)$, is a square.
    b) Show that the diagonals of $EFGH$ are perpendicular to each other.

14. The vertices of quadrilateral $PQRS$ are at $P(0, -5)$, $Q(-9, 2)$, $R(-5, 8)$, and $S(4, 2)$. Show that $PQRS$ is not a rectangle.

15. **C** A square is a special type of rectangle. A square is also a special type of rhombus. How would you apply these descriptions of a square when using the coordinates of the vertices of a quadrilateral to determine the type of quadrilateral? Include examples in your explanation.

16. Determine the type of quadrilateral described by each set of vertices. Give reasons for your answers.
    a) $J(-5, 2)$, $K(-1, 3)$, $L(-2, -1)$, $M(-6, -2)$
    b) $E(-5, -4)$, $F(-5, 1)$, $G(7, 4)$, $H(7, -1)$
    c) $D(-1, 3)$, $E(6, 4)$, $F(4, -1)$, $G(-3, -2)$
    d) $P(-5, 1)$, $Q(3, 3)$, $R(4, -1)$, $S(-4, -3)$

17. **A** A surveyor is marking the corners of a building lot. If the corners have coordinates $A(-5, 4)$, $B(4, 9)$, $C(9, 0)$, and $D(0, -5)$, what shape is the building lot? Include your calculations in your answer.

18. **T** Points $P(4, 12)$, $Q(9, 14)$, and $R(13, 4)$ are three vertices of a rectangle.
    a) Determine the coordinates of the fourth vertex, S.
    b) Briefly describe how you found the coordinates of S.
    c) Predict whether the lengths of the diagonals of rectangle $PQRS$ are the same length. Check your prediction.

19. Suppose that you know the coordinates of the vertices of a quadrilateral. What calculations would help you determine if the quadrilateral is a special type, such as a parallelogram, rectangle, rhombus, or square? How would you use the coordinates of the vertices in your calculations? Organize your thoughts in a flow chart.

Extending

20. a) Show that the midpoints of any pair of sides in a triangle are two of the vertices of another triangle, which has dimensions that are exactly one-half the dimensions of the original triangle and a side that is parallel to a side in the original triangle.
    b) Show that the midpoints of the sides in any quadrilateral are the vertices of a parallelogram.