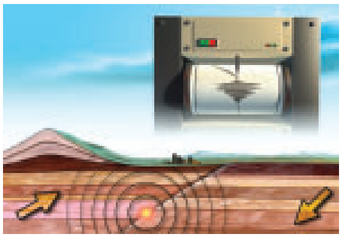
2.3 Equation of a Circle

YOU WILL NEED
- graphing calculator
- grid paper, ruler, and compass, or dynamic geometry software



Career Connection

A geologist studies the physical structure and processes of Earth. Professional geologists work for a wide range of government agencies, private firms, non-profit organizations, and academic instututions.

Tech Support

For help constructing a circle and plotting points using dynamic geometry software, see Appendix B-34 and B-18.

GOAL

Develop and use an equation for a circle.

INVESTIGATE the Math

When an earthquake occurs, seismographs can be used to record the shock waves. The shock waves are then used to locate the epicentre of the earthquake—the point on Earth located directly above the rock movement. The time lag between the shock waves is used to calculate the distance between the epicentre and each recording station, which avoids considering direction.

A seismograph in Collingwood, Ontario, recorded vibrations indicating that the epicentre of an earthquake was 30 km away.

? What equation describes the possible locations of the epicentre of the earthquake, if (0, 0) represents the location of the seismograph?

A. Tell why the equation of a circle that has its centre at the origin and a radius of 30 describes all the possible locations of the epicentre.

B. Sketch this circle on a grid. Then identify the coordinates of all its intercepts.

C. Show that (24, 18), (−24, 18), (24, −18), and (−24, −18) are possible locations of the epicentre, using
i) your graph **ii)** the distance formula

D. Let point $A(x, y)$ be any possible location of the epicentre. What is the length of OA? Explain.

E. Use the distance formula to write an expression for the length of OA.

F. Use your results for parts D and E to write an equation for the circle that is centred at the origin. Write your equation in a form that does not contain a square root. Explain why your equation describes all the possible locations of the epicentre.

Reflecting

G. If (x, y) is on the circle that is centred at the origin, so are $(x, -y)$, $(-x, -y)$, and $(-x, y)$. How does your equation show this?

H. How is the equation of a circle different from the equation of a linear relationship?

I. What is the equation of any circle that has its centre at the origin and a radius of r units?

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine the equation of a circle

A stone is dropped into a pond, creating a circular ripple. The radius of the ripple increases by 4 cm/s. Determine an equation that models the circular ripple, 10 s after the stone is dropped.

Aurora's Solution

$$x^2 + y^2 = r^2$$
$$r = (4 \text{ cm/s})(10 \text{ s})$$
$$r = 40 \text{ cm}$$

I named the point where the stone entered the water (0, 0). I knew that the equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

I wanted to determine the radius of the circle at 10 s, so I multiplied the rate at which the radius increases by 10.

$$x^2 + y^2 = 40^2$$
$$x^2 + y^2 = 1600$$

I substituted the value of the radius for r into the equation.

The equation of the circular ripple is $x^2 + y^2 = 1600$.

EXAMPLE 2 Selecting a strategy to graph a circle, given the equation of the circle

A circle is defined by the equation $x^2 + y^2 = 25$. Sketch a graph of this circle.

Francesco's Solution

$x^2 + y^2 = 25$

To determine the x-intercepts, let $y = 0$.

I decided to determine some points on the circle. I knew that I could determine the intercepts by setting each variable equal to 0 and solving for the other variable.

$$x^2 + 0^2 = 25$$
$$x^2 = 25$$
$$\sqrt{x^2} = \pm\sqrt{25}$$
$$x = \pm 5$$

I remembered that there are two possible square roots: one positive and one negative.

The x-intercepts are located at (5, 0) and $(-5, 0)$.

To determine the y-intercepts, let $x = 0$.

$$0^2 + y^2 = 25$$
$$y^2 = 25$$
$$\sqrt{y^2} = \pm\sqrt{25}$$
$$y = \pm 5$$

I solved for y.

The y-intercepts are located at $(0, 5)$ and $(0, -5)$.

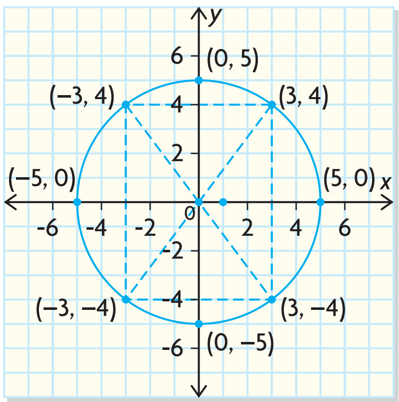
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$M_{AB} = \left(\frac{5 + (-5)}{2}, \frac{0 + 0}{2}\right)$$
$$= (0, 0)$$

The centre is at $(0, 0)$.

Since a circle has symmetry, I reasoned that the x-intercepts are endpoints of a diameter. Since all the points on a circle are the same distance from the centre, the midpoint of this diameter must be the centre.

The radius equals 5 units.

The line segments that join $(0, 0)$ to each intercept are radii. Since these are horizontal and vertical lines whose lengths are 5 units, this circle has a radius of 5 units.



I plotted the intercepts and then joined them with a smooth circle. I noticed that $(3, 4)$ is on the circle, and that points with similar coordinates are too. This makes sense because a circle with centre $(0, 0)$ is symmetrical about any line that passes through the origin.

EXAMPLE 3 **Reasoning** to determine the equation of a circle

A circle has its centre at $(0, 0)$ and passes through point $(8, -6)$.

a) Determine the equation of the circle.

b) Determine the other endpoint of the diameter through $(8, -6)$.

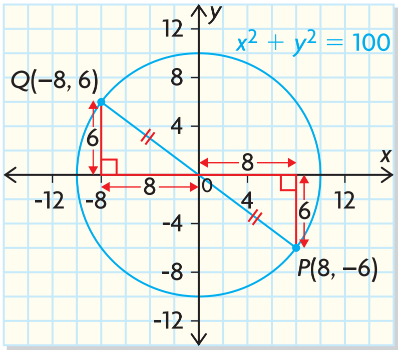
Trevor's Solution

a)
$$x^2 + y^2 = r^2$$
$$8^2 + (-6)^2 = r^2$$
$$100 = r^2$$

The equation of the circle is $x^2 + y^2 = 100$.

I started with the equation $x^2 + y^2 = r^2$, since the circle is centred at the origin. I knew that point $(8, -6)$ is on the circle, so I substituted 8 for x and -6 for y into the equation.

b) $r = \sqrt{100}$
$r = 10$



I calculated the radius of the circle. Then I drew a sketch, making sure that the circle passed through 10 and −10 on both the *x*- and *y*-axes since 10 is the radius.

I drew the diameter that starts at *P*(8, −6), passes through (0, 0), and ends at point *Q*.

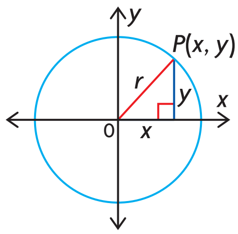
Because a circle is symmetrical and *PQ* is a diameter, I reasoned that *Q* has coordinates (−8, 6).

The other endpoint of the diameter that passes through point (8, −6) has coordinates (−8, 6).

In Summary

Key Idea

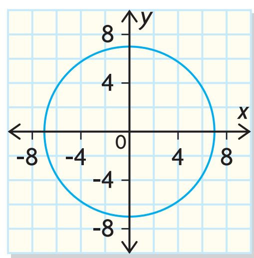
- Using the distance formula, you can show that the equation of a circle with centre (0, 0) and radius *r* is $x^2 + y^2 = r^2$.



Need to Know

- Every point on the circumference of a circle is the same distance from the centre of the circle.
- Once you know one point on a circle with centre (0, 0), you can determine other points on the circle using symmetry. If (x, y) is on a circle with centre (0, 0), then so are $(-x, y)$, $(-x, -y)$, and $(x, -y)$.

CHECK Your Understanding



1. The graph at the right shows a circle with its centre at (0, 0).
   a) State the *x*-intercepts of the circle.
   b) State the *y*-intercepts.
   c) State the radius.
   d) Write the equation of the circle.

2. Write the equation of a circle with centre (0, 0) and radius *r*.

   a) $r = 3$ **b)** $r = 50$ **c)** $r = 2\frac{1}{3}$ **d)** $r = 400$ **e)** $r = 0.25$

3. For each equation of a circle:
   i) Determine the radius.
   ii) State the x- and y-intercepts.
   iii) Sketch its graph.

   a) $x^2 + y^2 = 36$
   b) $x^2 + y^2 = 49$
   c) $x^2 + y^2 = 0.04$
   d) $x^2 + y^2 = 169$

PRACTISING

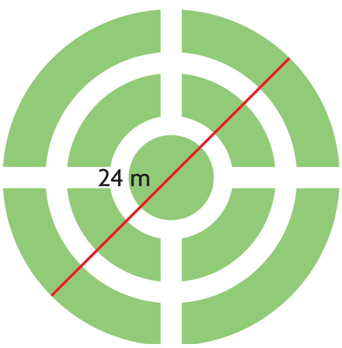
4. Use the given information to write an equation for a circle with centre (0, 0).
   a) radius of 11 units
   b) x-intercepts $(-9, 0)$ and $(9, 0)$
   c) y-intercepts $(0, -4)$ and $(0, 4)$
   d) diameter of 12 units

5. A circle has its centre at (0, 0) and passes through the point (8, 15).
   a) Calculate the radius of the circle.
   b) Write an equation for the circle.
   c) Sketch the graph.

6. Which of the following points are on the circle with equation $x^2 + y^2 = 65$? Explain.
   a) $(-4, 7)$ b) $(5, -6)$ c) $(8, -1)$ d) $(-3, -6)$

7. **K** a) Determine the radius of a circle that is centred at (0, 0) and passes through
   i) $(-3, 4)$ ii) $(5, 0)$ iii) $(0, -3)$ iv) $(8, -15)$
   b) Write the equation of each circle in part a).
   c) State the coordinates of two other points on each circle.
   d) Sketch the graph of each circle.

8. Write an equation for a circle that models each situation. Assume that (0, 0) is the centre of the circle in each situation.
   a) the possible locations of the epicentre of an earthquake, which is recorded to be a distance of 144 km from a seismograph station in Toronto
   b) the path of a satellite in a circular orbit at a distance of 19 000 km from the centre of Earth
   c) the rim of a bicycle wheel with a diameter of 69 cm
   d) the cross-section of a storm-water tunnel that has a diameter of 2.4 m



9. Don has designed a circular vegetable garden with a diameter of 24.0 m, as shown in the sketch at the left. He has included a circular flowerbed, 6.0 m in diameter, at the centre of the garden, as well as paths that are 1.5 m wide. Don needs to make a plan on grid paper for the landscape gardeners, who will create the garden. Determine the equations of all the circles he needs to draw. Assume that the centre of the garden is (0, 0) and all noncircular gardens are of equal width.

10. Two satellites are orbiting Earth. The path of one satellite has the
A equation $x^2 + y^2 = 56\ 250\ 000$. The orbit of the other satellite is 200 km farther from the centre of Earth. In one orbit, how much farther does the second satellite travel than the first satellite?

Career Connection

Engineers design satellites for communication including television, the Internet, and phone systems. Other uses of satellites include observation in the area of espionage, geology, and navigation such as GPS systems.

11. A circle has its centre at (0, 0) and passes through point $P(5, -12)$.

a) Determine the equation of the circle.

b) Determine the coordinates of the other endpoint of the diameter that passes through point P.

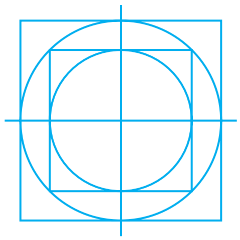
12. Determine the equation of a circle that has a diameter with endpoints $(-8, 15)$ and $(8, -15)$.

13. A rock is dropped into a pond, creating a circular ripple. The radius of the ripple increases steadily at 6 cm/s. A toy boat is floating on the pond, 2.00 m east and 1.00 m north of the spot where the rock is dropped. How long does the ripple take to reach the boat?

14. Points $(a, 5)$ and $(9, b)$ are on the circle $x^2 + y^2 = 125$. Determine the possible values of a and b. Round to one decimal place, if necessary.

15. A satellite orbits Earth on a path with $x^2 + y^2 = 45\ 000\ 000$.
C Another satellite, in the same plane, is currently located at (12 504, 16 050). Explain how you would determine whether the second satellite is inside or outside the orbit of the first satellite.

16. Chanelle is creating a design for vinyl flooring.
T She uses circles and squares to create the design, as shown. If the equation of the small circle is $x^2 + y^2 = 16$, what are the dimensions of the large square?



17. What reasons would you use to convince someone that it makes sense for the equation of a circle with centre (0, 0) and radius r to be $x^2 + y^2 = r^2$? Use as many reasons as you can.

Extending

18. Describe the circle with each equation.

a) $9x^2 + 9y^2 = 16$ **b)** $(x - 2)^2 + (y + 4)^2 = 9$

19. A truck with a wide load, proceeding slowly along a secondary road, is approaching a tunnel that is shaped like a semicircle. The maximum height of the tunnel is 5.25 m. If the load is 8 m wide and 3.5 m high, will it fit through the tunnel? Show your calculations, and explain your reasoning.