1.4 Solving Linear Systems: Substitution

GOAL

Solve a system of linear equations using an algebraic strategy.

YOU WILL NEED

- grid paper
- ruler
- graphing calculator

LEARN ABOUT the Math

Marla and Nancy played in a volleyball marathon for charity. They played for 38 h and raised \$412. Marla was sponsored for \$10/h. Nancy was sponsored for \$12/h.

? How many hours did each student play?

EXAMPLE 1 | **Selecting an algebraic strategy** to solve a linear system

Determine the length of time that each student played algebraically.

Isabel's Solution

Let m represent the hours Marla played. Let n represent the hours Nancy played.

I used variables for the number of hours each student played.

$m + n = 38$ ①
$10m + 12n = 412$ ②

I wrote one equation for the hours played and another equation for the money raised. I am looking for an ordered pair (m, n) that satisfies both equations.

$m = 38 - n$

I decided to solve for a variable in equation ① and then substitute into equation ②. I solved for m in equation ① since this was easier than solving for n.

$10(38 - n) + 12n = 412$

I used a **substitution strategy** by substituting the expression for m into equation ②. This gave me an equation in one variable, which included both pieces of information I had.

substitution strategy

a method in which a variable in one expression is replaced with an equivalent expression from another expression, when the value of the variable is the same in both

$$10(38) - 10n + 12n = 412$$
$$380 + 2n = 412$$
$$2n = 412 - 380$$
$$2n = 32$$
$$n = \frac{32}{2}$$
$$n = 16$$

I used the distributive property to multiply.

I solved for n using inverse operations.

$$m = 38 - n$$
$$m = 38 - 16$$
$$m = 22$$

I solved for m by substituting the value of n into the expression for m.

Marla played for 22 h and Nancy played for 16 h.

22 h + 16 h = 38h

22 h @ \$10/h = \$220

16 h @ \$12/h = \$192

\$220 + \$192 = \$412

They raised \$412.

I verified my solution. The total number of hours played by both girls and the total amount they raised matches the information in the problem.

Reflecting

A. When you substitute to solve a linear system, does it matter which equation or which variable you start with? Explain.

B. Why did Isabel need to do the second substitution after solving for n?

C. What would you do differently if you substituted for n instead of m?

APPLY the Math

EXAMPLE 2 Solving a problem modelled by a linear system using substitution

Most gold jewellery is actually a mixture of gold and copper. A jeweller is reworking a few pieces of old gold jewellery into a new necklace. Some of the jewellery is 84% gold by mass, and the rest is 75% gold by mass. The jeweller needs 15.00 g of 80% gold for the necklace. How much of each alloy should he use?

Wesley's Solution

Let x represent the mass of the 84% alloy in grams. Let y represent the mass of the 75% alloy in grams.

I used variables for the mass of each alloy.

$x + y = 15.00$ ①

I wrote an equation to represent the total mass of both alloys.

$0.84x + 0.75y = 0.80(15)$
$0.84x + 0.75y = 12.00$ ②

I wrote another equation to represent the amount of pure gold in the necklace, with the percents as decimals.
I calculated 80% of 15.00 as 12.00. This means that the final 15.00 g of the 80% alloy must contain 12.00 g of pure gold.

$y = 15.00 - x$

I solved for *y* in equation ①.

$0.84x + 0.75(15.00 - x) = 12.00$
$0.84x + 11.25 - 0.75x = 12.00$

I substituted 15.00 − *x* for *y* in equation ②. Then I used the distributive property to multiply.

$0.84x - 0.75x = 12.00 - 11.25$
$0.09x = 0.75$
$x = \frac{0.75}{0.09}$
$x \doteq 8.33$

I solved for *x*, the mass of the 84% alloy, to the nearest hundredth of a gram.

$y = 15.00 - x$
$y = 15.00 - 8.33$
$y = 6.67$

I substituted the value of *x* into *y* = 15.00 − *x* and calculated the mass of the 75% alloy to the nearest hundredth of a gram.

The jeweller should use about 8.33 g of the 84% alloy and about 6.67 g of the 75% alloy.

EXAMPLE 3 Connecting the solution to a linear system to the break-even point

Sarah is starting a business in which she will hem pants. Her start-up cost, to buy a sewing machine, is $1045. She will use about $0.50 in materials to hem each pair of pants. She will charge $10 for each pair of pants. How many pairs of pants does Sarah need to hem to break even?

Robin's Solution

Let x represent the number of pairs of pants that Sarah hems, let C represent her total costs, and let R represent her revenue.

I chose letters for the variables in this problem.

$C = 1045 + 0.50x$

The cost of materials to hem *x* pairs of pants is $0.50*x*. I added the start-up cost to get the total cost.

$R = 10x$

Sarah charges $10 for each pair of pants. When she hems *x* pairs of pants, her revenue is $10*x*.

Communication *Tip*

A company makes a profit when it has earned enough revenue from sales to pay its costs. The point at which revenue and costs are equal is the break-even point.

At the break-even point, total costs and total revenue are the same.

$y = 1045 + 0.50x$ ①
$y = 10x$ ②

Because costs and revenue are the same at the break-even point, I used y to represent both dollar amounts.

$10x = 1045 + 0.50x$

I substituted the expression for y in equation ② into equation ①.

$$10x - 0.50x = 1045$$
$$9.50x = 1045$$
$$x = \frac{1045}{9.50}$$
$$x = 110$$

I used inverse operations to solve for x.

$y = 10(110)$
$y = 1100$

I determined y by substituting the value of x into equation ②.

Check:

$C = 1045 + 0.50(110)$
$C = 1045 + 55$
$C = 1100$

$R = 10(110)$
$R = 1100$

I verified my solution. The break-even point is 110 pairs of pants since the cost and revenue are both $1100.

Sarah will break even when she has hemmed 110 pairs of pants.

EXAMPLE 4 **Selecting a substitution strategy** to solve a linear system

Determine, without graphing, where the lines with equations $5x + 2y = -2$ and $2x - 3y = -16$ intersect.

Carmen's Solution

$5x + 2y = -2$ ①
$2x - 3y = -16$ ②

$$2y = -2 - 5x$$
$$\frac{2y}{2} = \frac{-2 - 5x}{2}$$
$$y = -1 - \frac{5}{2}x \quad ③$$

I decided to isolate the variable y in equation ①. This resulted in an equivalent form of the equation, which I called equation ③.

$$2x - 3\left(-1 - \frac{5}{2}x\right) = -16$$

The values of *x* and *y* must satisfy both equations at the point of intersection, so I substituted the expression for *y* in equation ③ into equation ②.

$$2x + 3 + \frac{15}{2}x = -16$$
$$2x + \frac{15}{2}x = -16 - 3$$
$$2x + \frac{15}{2}x = -19$$

I used the distributive property to multiply, and then I simplified.

$$2(2x) + 2\left(\frac{15}{2}x\right) = 2(-19)$$
$$4x + 15x = -38$$
$$19x = -38$$
$$\frac{19x}{19} = \frac{-38}{19}$$
$$x = -2$$

I multiplied all the terms in the equation by the lowest common denominator of 2 to eliminate the fractions. Then I used inverse operations to solve for *x*.

$$y = -1 - \frac{5}{2}(-2)$$
$$y = -1 + \frac{10}{2}$$
$$y = -1 + 5$$
$$y = 4$$

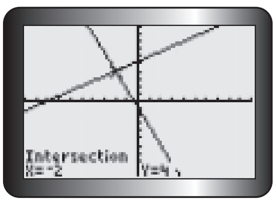
I substituted the value of *x* into equation ③. Then I determined the value of *y*.

The lines intersect at the point (–2, 4).

Check by graphing:

$$2x - 3y = -16 \quad ②$$
$$-3y = -2x - 16$$
$$y = \frac{2}{3}x + \frac{16}{3}$$

I solved for *y* in equation ② to get the equation in the form $y = mx + b$.



I graphed equations ① and ② using a graphing calculator. I used the Intersect operation to verify the point of intersection.

The graph confirms that the lines intersect at (−2, 4).

Tech Support

For help using a TI-83/84 graphing calculator to determine the point of intersection, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.

In Summary

Key Idea

- To determine the solution to a system of linear equations algebraically:
  - Isolate one variable in one of the equations.
  - Substitute the expression for this variable into the other equation.
  - Solve the resulting linear equation.
  - Substitute the solved value into one of the equations to determine the value of the other variable.

Need to Know

- Substitution is a convenient strategy when one of the equations can easily be rearranged to isolate a variable.
- Solving the equation created by substituting usually involves the distributive property. It may involve operations with fractional expressions.
- To verify a solution, you can use either of these strategies:
  - Substitute the solved values into the equation that you did not use when you substituted.
  - Graph both linear relations on a graphing calculator, and determine the point of intersection.

CHECK Your Understanding

1. For each equation, isolate the indicated variable.

a) $10x - y = 1, y$ **c)** $\frac{1}{2}x + y = 10, x$

b) $4x - y + 3 = 0, x$ **d)** $2x - y = 12, y$

2. To raise money for a local shelter, some Grade 10 students held a car wash and charged the prices at the left. They washed 53 vehicles and raised $382.

a) Write an equation to describe the number of vehicles washed.

b) Write an equation to describe the amount of money raised in terms of the number of each type of vehicle.

c) Solve for one of the variables in your equation for part a).

d) Substitute your expression for part c) into the equation for part b). Solve the new equation.

e) Substitute your answer for part d) into your equation for part a). Solve for the other variable. How many of each type of vehicle did the students wash?

PRACTISING

3. Decide which variable to isolate in one of the equations in each system. Then substitute for this variable in the other equation, and solve the system.

a) $x + 3y = 5$
$2x - 3y = -17$

b) $2x + y = 4$
$3x - 16y = 6$

4. Solve for the indicated variable.

a) $8a = 4 - b$, b

b) $6r + 3s = 9$, r

c) $3u + 7v = 21$, v

d) $0.3x - 0.3y = 1.8$, x

e) $0.12x - 0.06y = 0.24$, y

f) $\frac{1}{3}x + \frac{1}{2}y = 5$, x

5. Decide which variable to isolate. Then substitute for this variable, and solve the system.
K

a) $y = x - 5$, $x + y = 9$

b) $x = y + 4$, $3x + y = 16$

c) $x + 4y = 21$, $4x - 16 = y$

d) $3x - 2y = 10$, $x + 3y = 7$

e) $2x + y = 5$, $x - 3y = 13$

f) $x + 2y = 0$, $x - y = -4$

6. Tom pays a one-time registration charge and regular monthly fees to belong to a fitness club. After four months, he had paid \$420. After nine months, he had paid \$795. Determine the registration charge and the monthly fee.

7. A health-food company packs almond butter in jars. Some jars hold 250 g. Other jars hold 500 g. On Tuesday, the company packed 186.5 kg of almond butter in 511 jars. How many jars of each size did they pack?

8. The difference between two angles in a triangle is 11°. The sum of the same two angles is 77°. Determine the measures of all three angles in the triangle.

9. Solve each system. Check your answers.

a) $x + 3y = 7$ and $3x - 2y = -12$

b) $3 = 2a - b$ and $4a - 3b = 5$

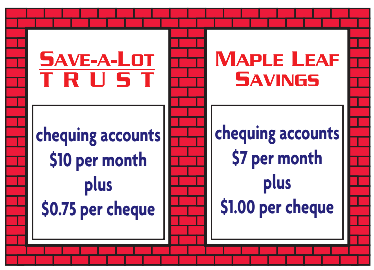
c) $7m + 2n = 21$ and $10m + 4n = -10$

d) $6x - 2y + 1 = 0$ and $3x - 5y + 7 = 0$

e) $3c + 2d = -24$ and $2c + 5d = -38$

f) $\frac{1}{4}x - 3y = \frac{1}{2}$ and $\frac{1}{3}x - 9y = 5$

10. Dan has saved \$500. He wants to open a chequing account at Save-A-Lot Trust or Maple Leaf Savings. Using the information at the right, which financial institution charges less per month?



11. Wayne wants to use a few pieces of silver to make a bracelet. Some of the jewellery is 80% silver, and the rest is 66% silver. Wayne needs 30.00 g of 70% silver for the bracelet. How much of each alloy should he use?

12. Sue is starting a lawn-cutting business. Her start-up cost to buy two
A lawn mowers and an edge trimmer is \$665. She has figured out that she will use about \$1 in gas for each lawn. If she charges \$20 per lawn, what will her break-even point be?

Safety *Connection*

Eye protection must be worn when operating a lathe.

13. A woodworking shop makes tables and chairs. To make a chair, 8 min is needed on the lathe and 8 min is needed on the sander. To make a table, 8 min is needed on the lathe and 20 min is needed on the sander. The lathe operator works 6 h/day, and the sander operator works 7 h/day. How many chairs and tables can they make in one day working at this capacity?

14. James researched these nutrition facts:
    - 1 g of soy milk has 0.005 g of carbohydrates and 0.030 g of protein.
    - 1 g of vegetables has 0.14 g of carbohydrates and 0.030 g of protein.

    James wants his lunch to have 50.000 g of carbohydrates and 20.000 g of protein. How many grams of soy milk and vegetables does he need?

15. Nicole has been offered a sales job at High Tech and a sales job at Best Computers. Which offer should Nicole accept? Explain.
    - High Tech: \$500 per week plus 5% commission
    - Best Computers: \$400 per week plus 7.5% commission

$$
\begin{aligned}
2x - (4x - 10) &= 4 \\
2x - 4x - 10 &= 4 \\
-2x - 10 &= 4 \\
-2x &= 14 \\
x &= -7 \\
y &= 4(-7) - 10 \\
y &= -28
\end{aligned}
$$

16. Monique solved the system of equations $2x - y = 4$ and $y = 4x - 10$
C by substitution. Her solution is at the left.
    a) What did she do incorrectly?
    b) Write a correct solution. Explain your steps.

17. Jennifer has nickels, dimes, and quarters in her piggy bank. In total,
T she has 49 coins, with a value of \$5.20. If she has five more dimes than all the nickels and quarters combined, how many of each type of coin does she have?

18. Explain why you think the strategy that was presented in this lesson is called substitution. Use the linear system $x + 4y = 8$ and $3x - 16y = 3$ in your explanation.

Extending

19. Marko invested \$300 000 in stocks, bonds, and a savings account at the rates shown at the left. He invested four times as much in stocks as he invested in the savings account. After one year, he earned \$35 600 in interest. How much money did he put into each type of investment?