Lesson 96 – Expected Value & Variance of Discrete Random Variables

HL2 Math - Santowski

Opening Exercise: Formulas

- If $\mu = \dfrac{\sum_{i=1}^{k} f_i x_i}{n}$ show that $\dfrac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n} = \dfrac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$

Expected Values of Discrete Random Variables

- The mean, or expected value, of a discrete random variable is

$$\mu = E(x) = \sum xp(x)$$

Variance of Discrete Random Variables

- The **variance** of a **discrete random variable** *x* is

$$\sigma^2 = E\left[(x-\mu)^2\right] = \sum (x-\mu)^2 p(x)$$

- The **standard deviation** of a **discrete random variable** *x* is

$$\sigma = \sqrt{\sigma^2}$$

Example #1

- You have a "weighted" coin that "favors" the outcome of heads such that p(H) = 0.6
- In our experiment, we will have two tosses of this coin and our DRV will be the number of times that heads appears
- Determine the probabilities of P(X = 0), P(X = 1) and P(X = 2) and then complete a distribution table and a probability histogram

Example #1

- You have a "weighted" coin that "favors" the outcome of heads such that p(H) = 0.6 In our experiment, we will have two tosses of this coin and our DRV will be the number of times that heads appears. Our distribution table looks like:

x	0	1	2
P(x)	0.16	0.48	0.36

- Now we want to calculate some "summary numbers" like→ expected value, variance & std deviation

Summary Measures Calculation Table

x	$p(x)$	$x\,p(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p(x)$
Total		$\Sigma x\,p(x)$ (expected value)			$\Sigma(x - \mu)^2\,p(x)$ (variance)

Summary Measures Calculation Table

x	$p(x)$	$x\,p(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 p(x)$
0	0.16	0	0-1.2=-1.2	1.44	0.2304
1	0.48	0.48	1-1.2=-0.2	0.04	0.0192
2	0.36	0.72	2-1.2=0.8	0.64	0.2304
Total		$\Sigma x\,p(x) = 1.2$		2.12	$\Sigma(x - \mu)^2\,p(x) = 0.48$

Example #1 - cont

- Now use the results from our calculation to confirm:

$$E\left[(X-\mu)^2\right] = E(X^2) - \left[E(X)\right]^2$$

Summary Measures

1. Expected Value (Mean of probability distribution)
   - Weighted average of all possible values
   - $\mu = E(x) = \Sigma x\, p(x)$
2. Variance
   - Weighted average of squared deviation about mean
   - $\sigma^2 = E[(x-\mu)^2] = \Sigma\,(x-\mu)^2\, p(x)$
3. Standard Deviation
   - $\sigma\sigma = \sqrt{\;^{2}}$

Expected Value

- Expected value is an extremely useful concept for good decision-making!

Example: the lottery

- The Lottery (also known as a tax on people who are bad at math…)
- A certain lottery works by picking 6 numbers from 1 to 49. It costs $1.00 to play the lottery, and if you win, you win $2 million after taxes.
- *If you play the lottery once, what are your expected winnings or losses?*

Lottery

Calculate the probability of winning in 1 try:

$$\frac{1}{\binom{49}{6}} = \frac{1}{\frac{49!}{43!6!}} = \frac{1}{13{,}983{,}816} = 7.2 \text{ x } 10^{-8}$$

"49 choose 6"

Out of 49 numbers, this is the number of distinct combinations of 6.

The probability function (note, sums to 1.0):

x$	*p(x)*
-1	.999999928
+ 2 million	*7.2 x* 10^{-8}

Expected Value

<u>The probability function</u>

x$	*p(x)*
-1	.999999928
+ 2 million	*7.2 x* 10^{-8}

<u>Expected Value</u>

E(X) = P(win)*$2,000,000 + P(lose)*-$1.00

= 2.0 *x* 10^6 * *7.2 x* 10^{-8} + *.999999928 (-1) = .144 - .999999928 = -$.86*

Negative expected value is never good!
You shouldn't play if you expect to lose money!

Expected Value

If you play the lottery every week for 10 years, what are your expected winnings or losses?

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520 x (-.86) = -$447.20

Gambling (or how casinos can afford to give so many free drinks…)

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet $1 that an odd number comes up, you win or lose $1 according to whether or not that event occurs. If random variable X denotes your net gain, determine the expected value of X.

Gambling (or how casinos can afford to give so many free drinks…)

A roulette wheel has the numbers 1 through 36, as well as 0 and 00. If you bet $1 that an odd number comes up, you win or lose $1 according to whether or not that event occurs. If random variable X denotes your net gain, determine the expected value of X.

$E(X) = 1(18/38) – 1\ (20/38) = -\$.053$

On average, the casino wins (and the player loses) 5 cents per game.

The casino rakes in even more if the stakes are higher:

$E(X) = 10(18/38) – 10\ (20/38) = -\$.53$

If the cost is $10 per game, the casino wins an average of 53 cents per game. If 10,000 games are played in a night, that's a cool $5300.

Practice Problem

On the roulette wheel, X=1 with probability 18/38 and X= -1 with probability 20/38.

We already calculated the mean to be = -$.053. What's the variance of X?

Answer

$$\sigma^2 = \sum_{\text{all } x} (x_i - \mu)^2 p(x_i)$$

$$= (+1 - -.053)^2 (18/38) + (-1 - -.053)^2 (20/38)$$
$$= (1.053)^2 (18/38) + (-1 + .053)^2 (20/38)$$
$$= (1.053)^2 (18/38) + (-.947)^2 (20/38)$$
$$= .997$$

$$\sigma = \sqrt{.997} = .99$$

Standard deviation is $.99. Interpretation: On average, you're either 1 dollar above or 1 dollar below the mean, which is just under zero. Makes sense!

Example 2

- Our HL Stats & Probability Unit test scores are described by the following probability distribution.

Score	40	50	60	70	80
P(Score)	.1	.2	.3	.3	.1

- Determine the mean and variance of the scores.
- Mr. S, in yet another act of benevolence, decides to scale the scores so my students will not be denied admission to the college of their choice. He decides the actual grades will become:
  - Grade = 1.5 * Score – 20 .
- Determine the mean and variance of the grades now.

Example 3

- A die is 'fixed' so that certain numbers will appear more often. The probability that a 6 appears is twice the probability of a 5 and 3 times the probability of a 4. The probabilities of 3, 2 and 1 are unchanged from a normal die.
- The probability distribution table is given below.

x	1	2	3	4	5	6
$\Pr(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{x}{3}$	$\frac{x}{2}$	x

- Find:
- (a) The value of x in the probability distribution and hence complete the probability distribution.
- (b) The probability of getting a 'double' with two of these dice. Compare with the 'normal' probability of getting a double

Example 4

Show that $p(x) = \dfrac{4x-3}{66}$, for x = 1,2, …6

is a probability distribution.

(a) State P(2 < x < 5)
(b) Determine E(X)

Example 5

- Consider the following gambling game, based on the outcome of the total of 2 dice:
- – if the total is a perfect square, you win $4
- – if the total is 2, 6, 8 or 10, you win $1
- – otherwise, you lose $2.
- (a) Find the expected value of this game.
- (b) Determine if it is a fair game.

Example 6

- Find the missing profit (or loss) so that the following probability table has an expected value of 0.

x	4	5	6	7	8	9	10
$\Pr(X = x)$	0.1	0.06	0.25	0.16	0.09	0.21	0.13
Gain	− 3	4	− 2	5	− 8	12	

Example 7

- For the following probability distribution calculate:
- $E(X)$
- $E(2X)$
- $E(X + 2)$
- $E(X^2)$
- $E(X^2) - [E(X)]^2$.

x	− 2	− 1	0	1	2	3	4	5
$\Pr(X = x)$	.05	.21	.12	.17	.2	.11	.07	.07

Examples

1. X is a discrete random variable:

x	1	2	3	4
$\mathbb{P}(X = x)$	k	$2k$	$3k$	$4k$

(a) Find the value of k.

(b) Find $\mathbb{E}(X)$.

(c) Find $\mathrm{Var}(X)$.

Examples - ANS

1. X is a discrete random variable:

x	1	2	3	4
$\mathbb{P}(X = x)$	k	$2k$	$3k$	$4k$

(a) Find the value of k. $\boxed{\frac{1}{10}}$

(b) Find $\mathbb{E}(X)$. $\boxed{3}$

(c) Find $\mathrm{Var}(X)$. $\boxed{1}$

Examples

2. Consider the probability distribution:

x	1	2	3	4
$\mathbb{P}(X = x)$	a	b	$\frac{1}{3}$	$\frac{1}{4}$

(a) By considering the probabilities, find an equation involving a and b.

(b) Given that $\mathbb{E}(X) = 2\frac{3}{4}$, find another equation involving a and b.

(c) Hence find a and b.

(d) Calculate $\text{Var}(X)$.

Examples - ANS

2. Consider the probability distribution:

x	1	2	3	4
$\mathbb{P}(X = x)$	a	b	$\frac{1}{3}$	$\frac{1}{4}$

(a) By considering the probabilities, find an equation involving a and b. $\boxed{\frac{5}{12} = a + b}$

(b) Given that $\mathbb{E}(X) = 2\frac{3}{4}$, find another equation involving a and b. $\boxed{\frac{3}{4} = a + 2b}$

(c) Hence find a and b. $\boxed{a = \frac{1}{12}, b = \frac{1}{3}}$

(d) Calculate $\text{Var}(X)$. $\boxed{\frac{41}{48}}$

Examples

5. A random number generator in a computer game produces values which can be modelled by the discrete random variable X with probability distribution given by

$$\mathbb{P}(X = x) = kx! \quad \text{for } x = 0, 1, 2, 3, 4$$
$$\mathbb{P}(X = x) = 0 \quad \text{otherwise.}$$

where k is a constant.

(a) Show that $k = \frac{1}{34}$ and illustrate the distribution with a sketch.

(b) Find the expectation and variance of X. $\boxed{\frac{7}{2}, \frac{61}{68}}$

Two independent values of X are generated. Let these values be X_1 and X_2.

(c) Show that $\mathbb{P}(X_1 = X_2)$ is a little more than $\frac{1}{2}$. $\boxed{\frac{309}{578}}$

(d) Given that $X_1 = X_2$, find the probability that X_1 and X_2 are each equal to 4. $\boxed{\frac{96}{103}}$

Examples

6. Two dice are rolled and the larger of the two scores is recorded (if they are the same, then the score of either dice is recorded). Calculate the probability of each possible score and calculate the expectation and variance of your recorded value. $\boxed{\frac{161}{36}, \frac{2555}{1296}}$

7. Two dice are rolled and the difference (always zero or a positive number) of the two scores is recorded (use D to represent this difference). Tabulate the possible scores with their probabilities. Calculate the expectation and variance of your recorded value. $\boxed{\frac{35}{18}, \frac{665}{324}}$