Lesson 81 – Working with systems of equations

HL2 Math - Santowski

Example #1

- Given the system $\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$
- Show that the solution to this system is

$$(x,y)=\left(\frac{ed-fb}{ad-bc},\frac{af-ec}{ad-bc}\right),\text{ where } ad-bc\neq 0$$

Example #1 (continued)

- Hence, or otherwise, solve the complex system

$$\begin{cases} 2x+(3-i)y=3 \\ ix+(1+2i)y=3i \end{cases}$$

Example #1 (continued)

- Hence, or otherwise, solve the complex system

$$\begin{cases} 2x+(3-i)y=3 \\ ix+(1+2i)y=3i \end{cases}$$

$$\text{ANS}: (x,y)=\left(\frac{1}{2}-\frac{1}{2}i,\ \frac{1}{2}+\frac{1}{2}i\right)$$

Example #2

- Solve, and then interpret, the system defined by

$$\begin{cases} 2x+4y+z=5 \\ 3x-5y-z=4 \\ x+y-z=6 \end{cases}$$

- Use the method of:
- (a) Substitution
- (b) elimination

Example #2

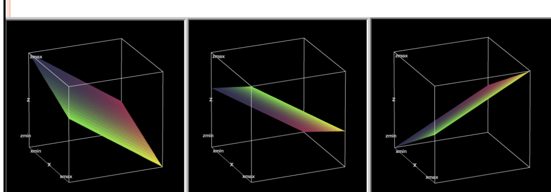
- Solve the system defined by

$$\begin{cases} 2x+4y+z=5 \\ 3x-5y-z=4 \\ x+y-z=6 \end{cases} \Rightarrow \quad \text{ANS}: (2,1,-3)$$

- Use the method of:
- (a) Substitution
- (b) elimination

Example #2 – Geometric Interpretation

- From http://www.math.uri.edu/~bkaskosz/flashmo/graph3d/



Solutions to Three Simultaneous Equations

- When solving systems where we have three variables (as in the geometric idea of having three planes in space), we have three possible scenarios:
- (a) one unique ordered triplet (a "point" that satisfies all three equations – a unique "intersection" point)
- (b) NO ordered triplet of real numbers that satisfies all three equations (no unique "intersection" point)
- (c) Infinitely many ordered triplets that satisfy all three equations

Example #3

- Use the Gaussian method of solving the simultaneous equations:

$$\begin{cases} x+3y-2z=3 \\ 2x-4y+3z=5 \\ 4x+y-z=6 \end{cases}$$

Example #3

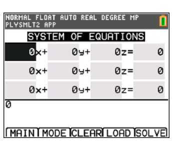
- Use the Gaussian method of solving the simultaneous equations:

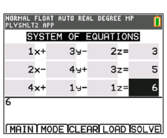
$$\begin{cases} x+3y-2z=3 \\ 2x-4y+3z=5 \\ 4x+y-z=6 \end{cases} \Rightarrow \quad (2,5,7)$$

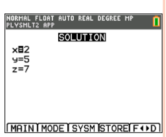
Example #3 – solution from TI-84

- Use the Gaussian method of solving the simultaneous equations:

$$\begin{cases} x+3y-2z=3 \\ 2x-4y+3z=5 \\ 4x+y-z=6 \end{cases}$$







Example #4

- Discuss all the possible types of solutions of this system of equations with respect to the real parameter ***K***.

$$\begin{cases} Kx+y+z=3 \\ x+y+z=1 \\ x+2y-z=2 \end{cases}$$

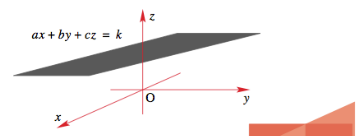
Example #4

- Discuss all the possible types of solutions of this system of equations with respect to the real parameter ***K***.

$$\begin{cases} Kx+y+z=3 \\ x+y+z=1 \\ x+2y-z=2 \end{cases} \Rightarrow (x,y,z)=\left(\frac{2}{K-1},\frac{3K-7}{3K-3},\frac{-2}{3K-3}\right)$$

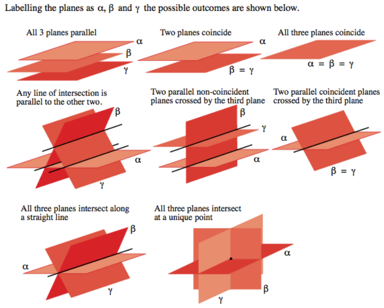
Geometric interpretations

Equations of the form $ax+by+cz=k$ represent a plane in space. To draw such a plane we need to set up three mutually perpendicular axes that coincide at some origin O. This is commonly drawn with a horizontal x–y plane and the z–axis in the vertical direction:



Geometric interpretations

There are a number of possible combinations for how three planes in space can intersect (or not). Labelling the planes as α, β and γ the possible outcomes are shown below.



Intersection of three planes – Case 1

Case (i)

When we write the equations of three planes such as :

$$x+y+2z=0 \quad (1)$$
$$2x-y+z=-6 \quad (2)$$
$$3x+4y-z=-6 \quad (3)$$

and consider their possible intersection, we are solving a system of equations in three unknowns, as already covered in Chapters 2 and 25. There are three possible outcomes:

(i) a single solution
(ii) no solution
(iii) an infinity of solutions.

Intersection of three planes – Case 1

Case (i)

When we write the equations of three planes such as :

$$x + y + 2z = 0 \quad (1)$$
$$2x - y + z = -6 \quad (2)$$
$$3x + 4y - z = -6 \quad (3)$$

and consider their possible intersection, we are solving a system of equations in three unknowns, as already covered in Chapters 2 and 25. There are three possible outcomes:

- **(i)** a single solution
- **(ii)** no solution
- **(iii)** an infinity of solutions.

To find this point we could eliminate z from (1) and (3), then from (2) and (3):

$$(1) + 2(3) \qquad 7x + 9y = -12$$
$$(2) + (3) \qquad 5x + 3y = -12$$

and then solve. We get $x = -3$ and $y = 1$, and by going back to (1) we find $z = 1$. Hence the point of intersection is $(-3, 1, 1)$.

(There is considerable freedom as to which variable to eliminate and how to set about eliminating it.)

Intersection of three planes – Case 2

Case (ii)

Now we look at a case where det M = 0 but there is no solution - i.e. the planes have no common point. Such a system is:

$$3x + y + 4z = 8 \quad (1)$$
$$3x - y - z = 4 \quad (2)$$
$$x + y + 3z = 2 \quad (3)$$

Intersection of three planes – Case 2

Case (ii)

Now we look at a case where det M = 0 but there is no solution - i.e. the planes have no common point. Such a system is:

$$3x + y + 4z = 8 \quad (1)$$
$$3x - y - z = 4 \quad (2)$$
$$x + y + 3z = 2 \quad (3)$$

We set off in the same way as in Case (i): by eliminating one of the variables in two different ways. For this system the obvious variable to eliminate is y:

$$(1) + (2) \qquad 6x + 3z = 12$$
$$(2) + (3) \qquad 4x + 2z = 6$$

The first equation is equivalent to $2x + z = 4$ and the second is equivalent to $2x + z = 3$. The equations are inconsistent with each other and there is no solution to the system. The three dimensional picture is of three planes that have no point of intersection.

Intersection of three planes – Case 3

Case (iii)

In this system check that det M = 0:

$$3x - y - z = 1 \quad (1)$$
$$x + 2y + z = 4 \quad (2)$$
$$x - 5y - 3z = -7 \quad (3)$$

Intersection of three planes – Case 3

Case (iii)

In this system check that det M = 0:

$$\begin{aligned} 3x-y-z &= 1 && (1)\\ x+2y+z &= 4 && (2)\\ x-5y-3z &= -7 && (3) \end{aligned}$$

It is important to be clear what this means: if we choose *any* y and z satisfying $7y+4z = 11$ we can find the value of x such that all three equations (1, 2 and 3) are satisfied. An example would be $y = z = 1$, leading to $x = 1$; check that all three equations are satisfied. But if we chose to

987

MATHEMATICS – Higher Level (Core)

satisfy $7y+4z = 11$ with $y = 5, z = -6$ we get $x = 0$, and again all three equations are satisfied. Clearly we could find as many solutions as we wanted. Solution is $\left(\frac{\lambda+6}{7}, \frac{11-4\lambda}{7}, \lambda\right)$.

Intersection of planes - practice

1. Solve the simultaneous equations

(a) $$\begin{aligned} 6x+4y-z &= 3\\ x+2y+4z &= -2\\ 5x+4y &= 0 \end{aligned}$$

(b) $$\begin{aligned} x+y+z &= 2\\ 4x+y &= 4\\ -x+3y+2z &= 8 \end{aligned}$$

(c) $$\begin{aligned} 4x+9y+13z &= 3\\ -x+3y+24z &= 17\\ 2x+6y+14z &= 6 \end{aligned}$$

(d) $$\begin{aligned} x-2y-3z &= 3\\ x+y-2z &= 7\\ 2x-3y-2z &= 0 \end{aligned}$$

(e) $$\begin{aligned} x-y-z &= 2\\ 3x+3y-7z &= 7\\ x+2y-3z &= 3 \end{aligned}$$

(f) $$\begin{aligned} x-2y &= -1\\ -x-y+3z &= 1\\ y-z &= 0 \end{aligned}$$

(g) $$\begin{aligned} x+y+z &= 1\\ x-y+z &= 3\\ 4x+2y+z &= 6 \end{aligned}$$

(h) $$\begin{aligned} -2x+y-2z &= 5\\ x+4z &= 1\\ x+y+10z &= 10 \end{aligned}$$