Lesson 80 - Mathematical Induction

HL2 - Santowski

Opening Exercise - Challenge

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- Take a pile of n stones
  - Split the pile into two smaller piles of size r and s
  - Repeat until you have n piles of 1 stone each
- Take the product of **all** the splits
  - So all the r's and s's from **each** split
- Sum up each of these products

Opening Exercise - An Example

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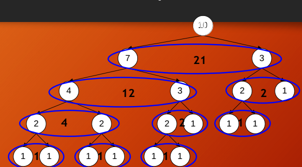

$$21+12+2+4+2+1+1+1+1+1=45=\frac{10*9}{2}$$

Opening Exercise - Challenge

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- Take a pile of n stones
  - Split the pile into two smaller piles of size r and s
  - Repeat until you have n piles of 1 stone each
- Take the product of **all** the splits
  - So all the r's and s's from **each** split
- Sum up each of these products
- Prove that this product equals $\frac{n(n-1)}{2}$

What is induction?

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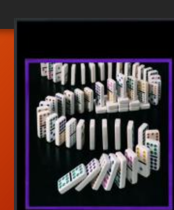
- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element



Induction example #1

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- Show that the sum of the first n odd integers is n^2

Induction example #1

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- Show that the sum of the first n odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show

$$\forall n \, \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i - 1 = n^2$$

- Base case: Show that P(1) is true

$$\mathrm{P}(1) = \sum_{i=1}^{1} 2(i) - 1 = 1^2$$
$$= 1 = 1$$

Induction example #1, continued

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- Inductive hypothesis: assume true for k
  - Thus, we assume that P(k) is true, or that

$$\boxed{\sum_{i=1}^{k}} 2i - 1 = \boxed{k}^2$$

  - Note: we don't yet know if this is true or not!
- Inductive step: show true for k+1
  - We want to show that:

$$\boxed{\sum_{i=1}^{k+1}} 2i - 1 = \boxed{(k+1)}^2$$

Induction example #1, continued

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- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i - 1 = k^2$
- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i - 1 = (k+1)^2$$
$$2(k+1) - 1 + \boxed{\sum_{i=1}^{k} 2i - 1} = k^2 + 2k + 1$$
$$2(k+1) - 1 \boxed{+ k^2} = k^2 + 2k + 1$$
$$k^2 + 2k + 1 = k^2 + 2k + 1$$

What did we show

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- Base case: P(1)
- If P(k) was true, then P(k+1) is true
  - i.e., P(k) → P(k+1)
- We know it's true for P(1)
- Because of P(k) → P(k+1), if it's true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it's true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it's true for P(3), then it's true for P(4)
- Because of P(k) → P(k+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity
- Thus, it is true for all possible values of n
- In other words, we showed that: $[\mathrm{P}(1) \wedge \forall k (\mathrm{P}(k) \to \mathrm{P}(k+1))] \to \forall n \, \mathrm{P}(n)$

The idea behind inductive proofs

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- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Why speling is not so important… 13

I cdnuolt blveieetaht I cluod aulaclty uesdnatnrd waht I was rdanieg. The phaonmneal pweor of thehmuan mind. Aoccdrnig to a rscheearch at Cmabrigde Uinervtisy, it deosn't mttaer in waht oredr the ltteers in a wrod are, the olny iprmoatnt tihng is taht thefrist and lsat ltteer be in the rghit pclae. The rset can be a taotl mses andyou can sitll raed it wouthit a porbelm. Tihs is bcuseae the huamn mnid deosnot raed ervey lteter by istlef, but the wrod as a wlohe. Amzanig huh? yaeh and I awlyas thought slpeling was ipmorantt.

Second induction example 14

- Example #2:
  - Show the sum of the first n positive even integers is $n^2 + n$

Second induction example 15

- Example #2:
  - Show the sum of the first n positive even integers is $n^2 + n$
  - Rephrased: $\forall n\ \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i = n^2 + n$
- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

Second induction example, continued 16

- Base case: Show P(1): $\mathrm{P}(1) = \sum_{i=1}^{1} 2(i) = 1^2 + 1$ $= 2 = 2$
- Inductive hypothesis: Assume $\mathrm{P}(k) = \sum_{i=1}^{k} 2i = k^2 + k$
- Inductive step: Show $\mathrm{P}(k+1) = \sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

Second induction example, continued 17

- Recall our inductive hypothesis: $\mathrm{P}(k) = \sum_{i=1}^{k} 2i = k^2 + k$

$$\sum_{i=1}^{k+1} 2i = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} = (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 = k^2 + 3k + 2$$

Notes on proofs by induction 18

- We manipulate the k+1 case to make part of it look like the k case
- We then replace that part with the other side of the k case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i = k^2 + k$$

$$\sum_{i=1}^{k+1} 2i = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} = (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} = (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 = k^2 + 3k + 2$$

Third induction example

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- Show $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Third induction example

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- Show $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- Base case: n = 1 $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$

$$1^2 = \frac{6}{6}$$

$$1 = 1$$

- Inductive hypothesis: assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

Third induction example

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- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6 \quad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

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- Show: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$

Third induction again: what if your inductive hypothesis was wrong?

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- Show: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$
- Base case: n = 1: $\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$

$$1^2 = \frac{7}{6}$$

$$1 \neq \frac{7}{6}$$

- But let's continue anyway…
- Inductive hypothesis: assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$

Third induction again: what if your inductive hypothesis was wrong?

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- Inductive step: show $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8 \quad \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

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- Show that $n! < n^n$ for all $n > 1$

Fourth induction example

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- Show that $n! < n^n$ for all $n > 1$
- Base case: $n = 2$
  $2! < 2^2$
  $2 < 4$
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

Fourth induction example

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- Show that $n! < n^n$ for all $n > 1$
- Base case: $n = 2$
  $2! < 2^2$
  $2 < 4$
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)\boxed{k!} < (k+1)\boxed{k^k} < (k+1)(k+1)^k = (k+1)^{k+1}$$

Proving De Moivre's Theorem by Induction

De Moivre's theorem:

$$(\cos A + i \sin A)^n = (\cos nA + i \sin nA)$$

Proof:

For $n = 1$

$(\cos A + \mathrm{i} \sin A)^1 = (\cos 1A + i \sin 1A)$. True where $n = 1$

Assume: true for $n = k$

$(\cos A + \mathrm{i} \sin A)^k = (\cos kA + i \sin kA)$.

Proving De Moivre's Theorem by Induction

To Prove: True for $n = k + 1$

$(\cos A + \mathrm{i} \sin A)^{k+1} = (\cos A + i \sin A)^k(\cos A + i \sin A)$

$= (\cos kA + i \sin kA)(\cos A + i \sin A)$ …. Assumed.

$= \cos(kA + A) + i \sin(kA + A)$

$= \cos(k + 1)A + i \sin(k + 1)A$

True for where $n = 1$

True for where $n = k + 1$ and so true for where $n = 2, 3, 4$ etc.

Further Examples - #1 & #2

1. Prove that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

2. Prove that $n(n+1)(n+2)$ is divisible by 6 for all n such that $n \in \mathbb{Z}^+$.

1. (i) Show statement true for $n=1$: $\sum_{r=1} r(r+1)=\frac{1}{3}(1)(1+1)(1+2)$

$\sum_{r=1}^{1} r(r+1)=1(1+1)=2$ and $\frac{1}{3}(1)(1+1)(1+2)=\frac{1}{3}(2)(3)=2$ $\Rightarrow$ statement true for $n=1$

(ii) Assume statement is true for a specific value of $n\in\mathbb{Z}^+$; i.e. statement true for some $n=k$:

that is, assume $\sum_{r=1}^{k} r(r+1)=\frac{1}{3}k(k+1)(k+2)$

(iii) Show that it must follow from this assumption that the statement is true for the next value of n; that is, show statement must be true for $n=k+1$:

$$\sum_{r=1}^{k+1} r(r+1)=\frac{1}{3}(k+1)(k+2)(k+3)$$

sum of $k+1$ terms = sum of k terms + value of term when $r=k+1$

$$\sum_{r=1}^{k} r(r+1)+(k+1)(k+2)=\frac{1}{3}(k+1)(k+2)(k+3)$$

$$\frac{1}{3}k(k+1)(k+2)+(k+1)(k+2)=\frac{1}{3}(k+1)(k+2)(k+3) \quad \text{[applying assumption from (ii)]}$$

$$(k+1)(k+2)\left[\frac{1}{3}k+1\right]=\frac{1}{3}(k+1)(k+2)(k+3)$$

$$\frac{1}{3}(k+1)(k+2)(k+3)=\frac{1}{3}(k+1)(k+2)(k+3) \qquad \textit{Q.E.D.}$$

The statement has been shown true for $n=1$ and given it's true for some $n=k, n\in\mathbb{Z}^+$ it has been shown that it follows that it must also be true for $n=k+1$; therefore, by mathematical induction the statement is true for all $n\in\mathbb{Z}^+$.

2. (i) Show statement true for $n=1$: $n(n+1)(n+2)=1(1+1)(1+2)=6$ and 6 is divisible by 6, thus statement is true for $n=1$.

(ii) Assume statement is true for a specific value of $n\in\mathbb{Z}^+$; i.e. statement true for some $n=k$: that is, assume $k(k+1)(k+2)$ is divisible 6; which means that $k(k+1)(k+2)$ must be equal to a multiple of 6; i.e. $k(k+1)(k+2)=6p$ where $p\in\mathbb{Z}^+$

(iii) Show that it must follow from this assumption that the statement is true for the next value of n; that is, show statement must be true for $n=k+1$:

need to show that $(k+1)(k+2)(k+3)$ is a multiple of 6

$k^3+6k^2+11k+6$ [note: $k(k+1)(k+2)=k^3+3k^2+2k$]

$(k^3+3k^2+2k)+3k^2+9k+6$

$6p+3(k+1)(k+2)$ [applying assumption from (ii)]

$(k+1)(k+2)$ is product of two consecutive integers – thus, one factor is even (a multiple of 2); therefore the product $(k+1)(k+2)$ must be a multiple of 2; i.e. $(k+1)(k+2)=2q,\ q\in\mathbb{Z}^+$

$$6p+3(k+1)(k+2)=6p+3\cdot 2q=6p+6q=6(p+q) \qquad \textit{Q.E.D.}$$

The statement has been shown true for $n=1$ and given true for some $n=k, n\in\mathbb{Z}^+$ it's been shown it follows that it must also be true for $n=k+1$; therefore, by mathematical induction the statement is true for all $n\in\mathbb{Z}^+$.

Further Examples - #3

3. (a) Find the first three derivatives of xe^{-x}.

(b) Suggest a formula for the nth derivative of xe^{-x}, that is $\frac{d}{dx}\left(xe^{-x}\right), n\in\mathbb{Z}^+$.

(c) Prove that your formula is true by mathematical induction.

(iii) Show that it must follow that the statement is true for the next value of n; that is, show statement must be true for $n=k+1$:

$$\frac{d^{k+1}}{dx^{k+1}}\left(xe^{-x}\right)=(-1)^{k+1}\left[x-(k+1)\right]e^{-x}$$

the $(k+1)$ derivative = derivative of the kth derivative

$$\frac{d}{dx}\left[\frac{d^k}{dx^k}\left(xe^{-x}\right)\right]=(-1)^{k+1}[x-k-1]e^{-x}$$

$$\frac{d}{dx}\left[(-1)^k(x-k)e^{-x}\right]=\text{RHS} \qquad \left[\text{applying assumption } \frac{d^k}{dx^k}\left(xe^{-x}\right)=(-1)^k(x-k)e^{-x}\right]$$

$$(-1)^k\left[\frac{d}{dx}\left(xe^{-x}-ke^{-x}\right)\right]=\text{RHS}$$

$$(-1)^k\left[\frac{d}{dx}\left(xe^{-x}\right)+\frac{d}{dx}\left(-ke^{-x}\right)\right]=\text{RHS}$$

$$(-1)^k\left[(1-x)e^{-x}+ke^{-x}\right]=\text{RHS}$$

$$(-1)^k\left[(-1)(x-k-1)e^{-x}\right]=\text{RHS}$$

$$(-1)^k(-1)\left[(x-k-1)e^{-x}\right]=\text{RHS}$$

$$(-1)^{k+1}[x-k-1]e^{-x}=(-1)^{k+1}[x-k-1]e^{-x} \qquad \textit{Q.E.D.}$$

The statement has been shown true for $n=1$ and given it's true for some $n=k, n\in\mathbb{Z}^+$ it has been shown that it follows that it must also be true for $n=k+1$; therefore, by mathematical induction the statement is true for all $n\in\mathbb{Z}^+$.

IB Examples

May 2005 Paper 1 TZ1

Using mathematical induction, prove that $\sum_{r=1}^{n}(r+1)2^{r-1}=n\left(2^n\right)$ for $n\in Z^+$

IB Examples

May 2007 Paper 2 TZ1

Prove by induction that $12^n+2\left(5^{n-1}\right)$ is a multiple of 7 for $n\in Z^+$

[Hint: For this one, you might want to call your multiples of 7 to be in the form of $7m$ for $m\in Z^+$]

IB Examples

May 2010 Paper 1 TZ1

(a) Show that $\sin 2nx = \sin((2n+1)x)\cos x - \cos((2n+1)x)\sin x$

(b) Hence prove, by induction that

$$\cos x + \cos 3x + \cos 5x + \ldots + \cos((2n-1)x) = \frac{\sin 2nx}{2x \sin x}$$

Mathematical Induction: Example

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- Show that any postage of ≥ 8¢ can be obtained using 3¢ and 5¢ stamps.
- First check for a few particular values:
  - 8¢ = 3¢ + 5¢
  - 9¢ = 3¢ + 3¢ + 3¢
  - 10¢ = 5¢ + 5¢
  - 11¢ = 5¢ + 3¢ + 3¢
  - 12¢ = 3¢ + 3¢ + 3¢ + 3¢
- How to generalize this?

Mathematical Induction: Example

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- Let P(n) be the sentence "n cents postage can be obtained using 3¢ and 5¢ stamps".
- Want to show that
  "P(k) is true" *implies* "P(k+1) is true"
  for any k ≥ 8¢.
- 2 cases: 1) P(k) is true and
  the k cents contain at least one 5¢.
  2) P(k) is true and
  the k cents do not contain any 5¢.

Mathematical Induction: Example

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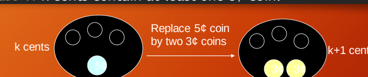
- *Case 1:* k cents contain at least one 5¢ coin.



- *Case 2:* k cents do not contain any 5¢ coin.
  Then there are at least three 3¢ coins.

Opening Exercise - Solution

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- We will show it is true for a pile of *k* stones, and show it is true for *k*+1 stones
  - So P(*k*) means that it is true for *k* stones
- Base case: *n* = 1
  - No splits necessary, so the sum of the products = 0
  - 1*(1-1)/2 = 0
  - Base case proven

Opening Exercise - Solution

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- Inductive hypothesis: assume that P(1), P(2), …, P(*k*) are all true
  - This is strong induction!
- Inductive step: Show that P(*k*+1) is true
  - We assume that we split the *k*+1 pile into a pile of *i* stones and a pile of *k*+1-*i* stones
  - Thus, we want to show that (*i*)*(*k*+1-*i*) + P(*i*) + P(*k*+1-*i*) = P(*k*+1)
  - Since 0 < *i* < *k*+1, both *i* and *k*+1-*i* are between 1 and *k*, inclusive

- Thus, we want to show that $(i)*(k+1-i) + P(i) + P(k+1-i) = P(k+1)$

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$$\mathrm{P}(i) = \frac{i^2 - i}{2}$$

$$\mathrm{P}(k+1-i) = \frac{(k+1-i)(k+1-i-1)}{2} = \frac{k^2 + k - 2ki - i + i^2}{2}$$

$$\mathrm{P}(k+1) = \frac{(k+1)(k+1-1)}{2} = \frac{k^2 + k}{2}$$

$$(i)*(k+1-i) + \mathrm{P}(i) + \mathrm{P}(k+1-i) = \mathrm{P}(k+1)$$

$$ki + i - i^2 + \frac{i^2 - i}{2} + \frac{k^2 + k - 2ki - i - i^2}{2} = \frac{k^2 + k}{2}$$

$$2ki + 2i - 2i^2 + i^2 - i + k^2 + k - 2ki - i + i^2 = k^2 + k$$

$$k^2 + k = k^2 + k$$