LESSON 79 – EXPONENTIAL FORMS OF COMPLEX NUMBERS

HL2 Math - Santowski

Opening Exercise #1

- List the first 8 terms of the Taylor series for (i) e^x, (ii) sin(x) and (iii) cos(x)
- HENCE, write the first 8 terms for the Taylor series for e^{θ}
- HENCE, write the first 8 terms for the Taylor series for $e^{i\theta}$. Simply your expression.
- What do you notice?

Opening Exercise #2

- If y(θ) = cos(θ) + ***i*** sin(θ), determine dy/d θ
- Simplify your expression and what do you notice?

EXPONENTIAL FORM

$$y(x) = \cos(x) + i\sin(x)$$

or

$$z(\theta) = \cos(\theta) + i\sin(\theta)$$

$$\frac{dy}{dx} = -\sin x + i\cos x \quad = i(i\sin x + \cos x)$$
$$= iy$$

Which function does this? $\Rightarrow$ $\frac{dy}{dx} = ky$

EXPONENTIAL FORM

$$y = Ae^{kx}$$

So (not proof but good enough!)

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

If you find this *incredible* or *bizarre*, it means you are paying attention.

Substituting $\theta = \pi$ gives "Euler's jewel":

$$\boxed{e^{i\pi} + 1 = 0}$$

which connects, simply, the 5 most important numbers in mathematics.

Forms of Complex Numbers

Forms of Complex Numbers

Rectangular		Polar		Exponential
$a + bj$	=	$r(\cos\theta + j\sin\theta)$	=	$re^{j\theta}$
		where $r = \sqrt{a^2 + b^2}$ and θ is computed from $\tan\theta = \frac{b}{a}, a \neq 0$		θ is in radians

Conversion Examples

- Convert the following complex numbers to all 3 forms:

(a) $4 - 4i$

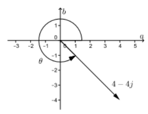
(b) $-2\sqrt{2} + 3\sqrt{2}i$

Example #1 - Solution

Example 1: Change $4-4j$ to polar and exponential forms.

Method

1. Draw a graphical representation of the complex number. **Use this to help determine the correct value for θ in the third step.**



2. Compute r. $\quad r=\sqrt{4^2+(-4)^2}=\sqrt{16+16}=4\sqrt{2}$

3. Compute θ. $\quad \tan\theta=\frac{-4}{4}=-1 \;\rightarrow\; \theta=315°=\frac{7\pi}{4}$ rad (fourth quadrant)

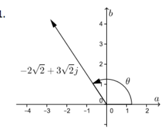
Solution

Rectangular form		**Polar form**		**Exponential form**
$4-4j$	=	$4\sqrt{2}(\cos 315° + j\sin 315°)$	=	$4\sqrt{2}\,e^{(7\pi/4)j}$
		or		
		$4\sqrt{2}\left(\cos\frac{7\pi}{4}\text{ rad} + j\sin\frac{7\pi}{4}\text{ rad}\right)$		

Example #2 - Solution

Example 2: Change $-2\sqrt{2}+3\sqrt{2}\,j$ to polar and exponential forms.

1.



2.
$$r=\sqrt{\left(-2\sqrt{2}\right)^2+\left(3\sqrt{2}\right)^2}$$
$$=\sqrt{8+18}=\sqrt{26}$$

3.
$$\tan\theta=\frac{3\sqrt{2}}{-2\sqrt{2}}=-1.5$$
$$\theta\approx 123.7°\approx 2.16\text{ rad}$$

Solution

Rectangular form		**Polar form**		**Exponential form**
$-2\sqrt{2}+3\sqrt{2}\,j$	=	$\sqrt{26}(\cos 123.7° + j\sin 123.7°)$	=	$\sqrt{26}\,e^{2.16j}$
		or		
		$\sqrt{26}(\cos 2.16\text{ rad} + j\sin 2.16\text{ rad})$		

Further Practice

Problems to Try

Change to polar and exponential forms:

1. $-3-2j$
2. $2\sqrt{3}-2j$

Change to rectangular form:

3. $15(\cos 120° + j\sin 120°)$
4. $12\left(\cos\frac{\pi}{3}+j\sin\frac{\pi}{3}\right)$
5. $6\,e^{(4\pi/3)j}$

Further Practice - Answers

Answers

1. $\sqrt{13}(\cos 213.7° + j\sin 213.7°)$ or $\sqrt{13}(\cos 3.73\text{ rad} + j\sin 3.73\text{ rad})$; $\sqrt{13}\,e^{3.73j}$
2. $4(\cos 330° + j\sin 330°)$ *or* $4\left(\cos\frac{11\pi}{6}+j\sin\frac{11\pi}{6}\right)$; $4\,e^{(11\pi/6)j}$
3. $-\frac{15}{2}+\frac{15\sqrt{3}}{2}j \;\approx\; -7.5+13.0j$
4. $6+6\sqrt{3}\,j \;\approx\; 6+10.4j$
5. $-3-3\sqrt{3}\,j \;\approx\; -3-5.20j$

Example 5

Example 5

Find complex number expressions, in Cartesian form, for

(a) $e^{i\pi/4}$ (b) e^{-i} (c) $e^{i\pi}$

Example 5 - Solutions

Solution

(a) $e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(b) $e^{-i} = \cos(-1) + i\sin(-1) = 0.540 - i(0.841)$

(c) $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

Verifying Rules …..

- If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$
- Use the exponential form of complex numbers to determine: (a) $z_1 \times z_2$

  (b) $\frac{z_1}{z_2}$

  (c) $\frac{1}{z_1}$

Example 6

Example 6

If $z = re^{i\theta}$ and $w = te^{i\phi}$ then find expressions for (a) z^{-1} (b) z^* (c) zw

Example 6 - Solutions

Solution

(a) If $z = re^{i\theta}$ then $z^{-1} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$ using the normal rules for indices.

(b) Working in polar form: if $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ then

$$z^* = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. In fact this reflects the general rule: to find the complex conjugate of any expression simply replace i by $-$i wherever it occurs in the expression.

(c) $zw = (re^{i\theta})(te^{i\phi}) = rte^{i\theta}e^{i\phi} = rte^{i\theta+i\phi} = rte^{i(\theta+\phi)}$ which is again the result we are familiar with: when complex numbers are multiplied their moduli multiply and their arguments add.

And as for powers …..

- Use your knowledge of exponent laws to simplify:

(a) $\left(5x^{3k}\right)^4$

And as for powers …..

- Use your knowledge of exponent laws to simplify:

(a) $\left(5x^{3k}\right)^4$

(b) $\left(5e^{3i}\right)^4$

So .. Thus …..

$$z^n = (rcis\theta)^n = r^n cis(n\theta)$$

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}$$

And de Moivres Theorem …..

- Use the exponential form to:

Example 2. Express:

a) $(1+i)^6$ in Cartesian form;

b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

de Moivres Theorem

Solution.

a) Change to polar form, use (8), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i6\pi/4} = 8e^{i3\pi/2} = -8i.$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (7).

Famous Equations …..

- Show that $e^{i\pi}+1=0$

Challenge Q

- Show that $(3+4i)^{(6+7i)} = -10.3 - 21.4i$