Lesson 76 – Introduction to Complex Numbers

HL2 MATH - SANTOWSKI

Lesson Objectives

(1) Introduce the idea of imaginary and complex numbers
(2) Practice operations with complex numbers
(3) Use complex numbers to solve polynomials
(4) geometric representation of complex numbers

To see a complex number we have to first see where it shows up

Solve both of these

$$x^2 - 81 = 0 \qquad x^2 + 81 = 0$$

$$x^2 = 81 \qquad x^2 = -81$$

$$x = \pm 9$$

Uhoh.......what do I do here?

Um, no solution????

$x = \pm\sqrt{-81}$ does not have a real answer.

It has an "imaginary" answer.

To define a complex number we have to create a new variable.

This new variable is " i "

Imaginary Unit

Until now, you have always been told that you can't take the square root of a negative number. If you use imaginary units, you can!

The imaginary unit is **i** where $i = \sqrt{-1}$

It is used to write the square root of a negative number.

Property of the square root of negative numbers

If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$

Examples:

$\sqrt{-3} = i\sqrt{3}$ $\qquad \sqrt{-4} = i\sqrt{4} = 2i$

Definition: $i = \sqrt{-1}$

Note: i is the representation for $\sqrt{-1}$, **not** a simplification of $\sqrt{-1}$

So, following this definition: $i^2 = -1$

So what is i^3 and i^4 ?

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So, following this definition: $i^2 = -1$

So $i^3 = -i$ and $i^4 = 1$

And it cycles….

$i = \sqrt{-1}$	$i^5 = i^4 \cdot i = i$	$i^9 = i^8 \cdot i = i$
$i^2 = -1$	$i^6 = i^4 \cdot i^2 = -1$	$i^{10} = i^8 \cdot i^2 = -1$
$i^3 = -i$	$i^7 = i^4 \cdot i^3 = -i$	$i^{11} = i^8 \cdot i^3 = -i$
$i^4 = 1$	$i^8 = 1^4 \cdot i^4 = 1$	$i^{12} = 1^8 \cdot i^4 = 1$

Do you see a pattern yet?

What is that pattern?

We are looking at the remainder when the power is divided by 4.

Why?

Every i^4 doesn't matter. It is what remains after all of the i^4 are taken out.

Try it with i^{92233}

Integral powers of i(iota)

$i^0 = 1$ (as usual)

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 . i = -i$

$i^4 = i^3 . i = -i . i = 1$

$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = -i$

$i^{-2} = \frac{1}{i^2} = -1$

$i^{-3} = \frac{1}{i^3} = \frac{1}{-i} = i$

$i^{-4} = \frac{1}{i^4} = 1$

Evaluate:

$$\left(i^{17} - \left(\frac{2}{i}\right)^3\right)^3$$

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Evaluate:

$$\left(i^{17} - \left(\frac{2}{i}\right)^3\right)^3$$

Solution

$$\left(i^{16} . i - \frac{8}{i^3}\right)^3 = \left(i + \frac{8}{i}\right)^3 = (i - 8i)^3$$

Ans: 343i

Illustrative Problem

If p,q,r, s are four consecutive integers, then $i^p + i^q + i^r + i^s =$

a)1 b) 2

c) 4 d) None of these

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c) 4 d) None of these

Solution: Note $q = p + 1$, $r = p + 2$, $s = p + 3$

Given expression $= i^p(1 + i + i^2 + i^3)$

$= i^p(1 + i - 1 - i) = 0$ Remember this.

Illustrative Problem

If $u_{n+1} = i\,u_n + 1$, where $u_1 = i + 1$, then u_{27} is

a) i b) 1

c) i + 1 d) 0

Illustrative Problem

If $u_{n+1} = i\,u_n + 1$, where $u_1 = i + 1$, then u_{27} is

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c) i + 1 d) 0

Solution: $u_2 = iu_1 + 1 = i(i+1) + 1 = i^2 + i + 1$

$u_3 = iu_2 + 1 = i(i^2+i+1) + 1 = i^3 + i^2 + i + 1$

Hence $u_n = i^n + i^{n-1} + \ldots + i + 1$

$u_{27} = i^{27} + i^{26} + \ldots + i + 1 = \frac{i^{28} - 1}{i - 1} = 0$

Note by previous question: $u_{27} = 0$

Hints to deal with *i*

1. Find all "*i*"s at the beginning of a problem.
2. Treat all "*i*"s like variables, with all rules of exponents holding.
3. Reduce the power of ***i*** at the end by the rules we just learned..

Examples

1. $\sqrt{-36} \cdot \sqrt{-81}$
2. $\sqrt{-36} + \sqrt{-81}$

COMPLEX NUMBERS

But what is $1+3i$

The two types of number (1 and 3*i*) cannot be "mixed".

Numbers of the form $k \times i$, $k \in$ ˆ are called imaginary numbers (or "pure imaginary")

Numbers like 1, 2, -3.8 that we used before are called real numbers.

When we combine them together in a sum we have complex numbers.

OK, so what is a complex number?

A complex number has two parts – a real part and an imaginary part.

A complex number comes in the form a + bi

real imaginary

COMPLEX NUMBERS

To summarize,

$$z = a + bi$$

- a and b are real numbers
- a is the "real part" of z; Re(z)
- b is the "imaginary part" of z; Im(z)
- The sum of the two parts is called a "complex number"

And just so you know…

All real numbers are complex → 3 = 3 + 0i

All imaginary numbers are complex → 7i = 0 + 7i

Again, treat the i as a variable and you will have no problems.

COMPLEX NUMBERS

Adding and subtracting complex numbers:

$z_1 = (2+3i)$

$z_2 = (4-9i)$

COMPLEX NUMBERS

Adding and subtracting complex numbers:

$z_1 = (2+3i)$

$z_2 = (4-9i)$

$z_1 + z_2 = \quad 6-6i$

$$(a+bi) \pm (c+di) \equiv (a \pm c) + (b \pm d)i$$

For addition and subtraction the real and imaginary parts are kept separate.

Adding and Subtracting
(add or subtract the real parts, then add or subtract the imaginary parts)

Ex: $(-1+2i)+(3+3i)$

Ex: $2i-(3+i)+(2-3i)$

Ex: $(2-3i)-(3-7i)$

Adding and Subtracting
(add or subtract the real parts, then add or subtract the imaginary parts)

Ex: $(-1+2i)+(3+3i)$
$=(-1+3)+(2i+3i)$
$=2+5i$

Ex: $2i-(3+i)+(2-3i)$
$=(-3+2)+(2i-i-3i)$
$=-1-2i$

Ex: $(2-3i)-(3-7i)$
$=(2-3)+(-3i--7i)$
$=-1+4i$

COMPLEX NUMBERS

Multiplying and dividing complex numbers:

$z_1=(2+3i)$
$z_2=(4-9i)$

COMPLEX NUMBERS

Multiplying and dividing complex numbers:

$z_1=(2+3i)$
$z_2=(4-9i)$

$$z_1z_2 = (2+3i)\times(4-9i)$$
$$=2\times4+(2\times-9i)+(3i\times4)+(3i\times-9i)$$
$$=8-18i+12i+(-27\times i^2)$$
$$=35-6i$$

$$(a+bi)\times(c+di)\equiv(ac-bd)+(bc+ad)i$$

Notice how, for multiplication, the real and imaginary parts "mix" through the formula $i^2=-1$.

Multiplying

Ex: $-i(3+i)$

Ex: $(2+3i)(-6-2i)$

Multiplying ➔ Treat the i's like variables, then change any that are not to the first power

Ex: $-i(3+i)$

$= -3i - i^2$

$= -3i - (-1)$

$= 1 - 3i$

Ex: $(2+3i)(-6-2i)$

$= -12 - 4i - 18i - 6i^2$

$= -12 - 22i - 6(-1)$

$= -12 - 22i + 6$

$= -6 - 22i$

COMPLEX CONJUGATES

What are the solutions to $x^2 - 6x + 21 = 0$?

$3 \pm 2\sqrt{3}i$

If we write $z = 3 + 2\sqrt{3}i$

Then the complex conjugate is written as $z^* = 3 - 2\sqrt{3}i$ (* means conjugate)

Calculate the following:

$z + z^*$

$z - z^*$

zz^*

COMPLEX CONJUGATES

What are the solutions to $x^2 - 6x + 21 = 0$?

$3 \pm 2\sqrt{3}i$

If we write $z = 3 + 2\sqrt{3}i$

Then the complex conjugate is written as $z^* = 3 - 2\sqrt{3}i$ (* means conjugate)

Calculate the following:

$z + z^* = 6 = 2\,\mathrm{Re}(z)$

$z - z^* = 4\sqrt{3}i = 2\,\mathrm{Im}(z)$

$zz^* = 3^2 + \left(2\sqrt{3}\right)^2 = 21 = |z|^2$

COMPLEX NUMBERS

Dividing complex numbers:

$z_1 = (2+3i)$ $\quad \frac{z_1}{z_2} =$

$z_2 = (4-9i)$

COMPLEX NUMBERS

Dividing complex numbers:

$z_1 = (2+3i)$

$z_2 = (4-9i)$

$$\frac{z_1}{z_2} = \frac{(2+3i)}{(4-9i)}$$

$$= \frac{(2+3i)}{(4-9i)} \times \frac{(4+9i)}{(4+9i)}$$

$$= \frac{8+18i+12i+(27\times i^2)}{4\times 4+36i-36i+(-9\times 9\times i^2)}$$

$$= \frac{-19+30i}{97} = -\frac{19}{97}+\frac{30}{97}i$$

Remember this trick!!

$$Ex: \frac{3+11i}{-1-2i}$$

$$Ex: \frac{3+11i}{-1-2i} * \frac{-1+2i}{-1+2i}$$

$$= \frac{(3+11i)(-1+2i)}{(-1-2i)(-1+2i)}$$

$$= \frac{-3+6i-11i+22i^2}{1-2i+2i-4i^2}$$

$$= \frac{-3-5i+22(-1)}{1-4(-1)}$$

$$= \frac{-3-5i-22}{1+4}$$

$$= \frac{-25-5i}{5}$$

$$= \frac{-25}{5} - \frac{5i}{5}$$

$$= -5-i$$

More Practice

5. $6i^{-5}$

6. $\frac{6-i}{4}+\frac{4+2i}{3+i}$

Absolute Value of a Complex Number

The distance the complex number is from the origin on the complex plane.

If you have a complex number $(a+bi)$

the absolute value can be found using: $\sqrt{a^2+b^2}$

Examples

1. $|-2+5i|$

2. $|-6i|$

Examples

1. $|-2+5i|$

$=\sqrt{(-2)^2+(5)^2}$

$=\sqrt{4+25}$

$=\sqrt{29}$

2. $|-6i|$

$=\sqrt{(0)^2+(-6)^2}$

$=\sqrt{0+36}$

$=\sqrt{36}$

$=6$

Which of these 2 complex numbers is closest to the origin? -2+5i

Complex Conjugates Theorem

Roots/Zeros that are not *Real* are *Complex* with an *Imaginary* component. Complex roots with Imaginary components always exist in *Conjugate Pairs*.

If $a + bi$ $(b \neq 0)$ is a zero of a polynomial function, then its Conjugate, $a - bi$, is also a zero of the function.

Find Roots/Zeros of a Polynomial

If the known root is *imaginary*, we can use the *Complex Conjugates Theorem*.

Ex: Find all the roots of $f(x) = x^3 - 5x^2 - 7x + 51$

If one root is $4 - i$.

Because of the Complex Conjugate Theorem, we know that *another* root must be $4 + i$.

Can the third root also be imaginary?

Example (con't)

Ex: Find all the roots of $f(x) = x^3 - 5x^2 - 7x + 51$

If one root is $4 - i$.

If one root is $4 - i$, then one factor is $[x - (4 - i)]$, and

Another root is $4 + i$, & another factor is $[x - (4 + i)]$.

Multiply these factors:

$$\begin{aligned}[x-(4-i)]\ [x-(4+i)] &= x^2 - x(4+i) - x(4+i) + (4-i)(4+i) \\ &= x^2 - 4x - xi - 4x + xi + 16 - i^2 \\ &= x^2 - 8x + 16 - (-1) \\ &= x^2 - 8x + 17\end{aligned}$$

Example (con't)

Ex: Find all the roots of $f(x) = x^3 - 5x^2 - 7x + 51$

If one root is $4 - i$.

If the product of the two non-real factors is $x^2 - 8x + 17$ then the third factor (that gives us the neg. real root) is the quotient of $P(x)$ divided by $x^2 - 8x + 17$:

$$\begin{array}{r} x + 3 \\ x^2 - 8x + 17 \overline{\big) x^3 - 5x^2 - 7x + 51} \\ \underline{x^3 - 5x^2 - 7x + 51} \\ 0 \end{array}$$

The third root is x = -3

Now write a polynomial function of least degree that has **real coefficients**, a **leading coeff. of 1** and **1, -2+i, -2-i as zeros.**

Now write a polynomial function of least degree that has **real coefficients**, a **leading coeff. of 1** and **1, -2+i, -2-i as zeros.**

f(x)= (x-1)(x-(-2+i))(x-(-2-i))

f(x)= (x-1)(x+2 - i)(x+2+ i)

f(x)= (x-1)[(x+2) - i] [(x+2)+i]

f(x)= (x-1)[(x+2)^2 - i^2] Foil

f(x)=(x-1)(x^2 + 4x + 4 – (-1)) Take care of i^2

f(x)= (x-1)(x^2 + 4x + 4 + 1)

f(x)= (x-1)(x^2 + 4x + 5) Multiply

f(x)= x^3 + 4x^2 + 5x – x^2 – 4x – 5

f(x)= x^3 + 3x^2 + x - 5

Now write a polynomial function of least degree that has **real coefficients**, a **leading coeff. of 1** and **1, -2+i, -2-i as zeros.**

Now write a polynomial function of least degree that has **real coefficients**, a **leading coeff. of 1** and **4, 4, 2+i as zeros.**

Note: 2+i means 2 – i is also a zero

F(x)= (x-4)(x-4)(x-(2+i))(x-(2-i))

F(x)= (x-4)(x-4)(x-2-i)(x-2+i)

F(x)= (x^2 – 8x +16)[(x-2) – i][(x-2)+i]

F(x)= (x^2 – 8x +16)[(x-2)^2 – i^2]

F(x)= (x^2 – 8x +16)(x^2 – 4x + 4 – (– 1))

F(x)= (x^2 – 8x +16)(x^2 – 4x + 5)

F(x)= x^4– 4x^3+5x^2 – 8x^3+32x^2 – 40x+16x^2 – 64x+80

F(x)= x^4-12x^3+53x^2-104x+80

Further Examples

EXAMPLES: Find a polynomial with the given zeros

-1, -1, 3i, -3i

2, 4 + i, 4 – i

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0$.

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Solve: $x^4 - 6x^2 - 8x + 24 = 0$.

Solution Now we can solve the original equation as follows.

$x^4 - 6x^2 + 8x + 24 = 0$ **This is the given equation.**

$(x - 2)(x - 2)(x^2 + 4x + 6) = 0$ **This was obtained from the second synthetic division.**

$x - 2 = 0$ or $x - 2 = 0$ or $x^2 + 4x + 6 = 0$ **Set each factor equal to zero.**

$x = 2$ $x = 2$ $x^2 + 4x + 6 = 0$ **Solve.**

EXAMPLE: Solving a Polynomial Equation

Solve: $x^4 - 6x^2 - 8x + 24 = 0$.

Solution We can use the quadratic formula to solve $x^2 + 4x + 6 = 0$.

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ **We use the quadratic formula because $x^2 + 4x + 6 = 0$ cannot be factored.**

$= \frac{-4 \pm \sqrt{4^2 - 4(1)(6)}}{2(1)}$ **Let $a = 1$, $b = 4$, and $c = 6$.**

$= \frac{-4 \pm \sqrt{-8}}{2}$ **Multiply and subtract under the radical.**

$= \frac{-4 \pm 2i\sqrt{2}}{2}$ $\sqrt{-8} = \sqrt{4(2)(-1)} = 2i\sqrt{2}$

$= -2 \pm i\sqrt{2}$ **Simplify.**

The solution set of the original equation is {2, $-2 - i$, $-2 + i$ $i\sqrt{2}$, $i\sqrt{2}$

FIND ALL THE ZEROS

$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$

(Given that 1 + 3i is a zero of f)

$f(x) = x^3 - 7x^2 - x + 87$

(Given that 5 + 2i is a zero of f)

More Finding of Zeros

$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$

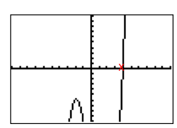
$f(x) = 3x^3 - 4x^2 + 8x + 8$

Find the zeros of $f(x) = x^3 - 11x - 20$

Hint: 4 is a zero

Find the zeros of $f(x) = x^3 - 11x - 20$

Hint: 4 is a zero



4	1	0	-11	-20
		4	16	20
	1	4	5	0

$(x-4)(x^2+4x+5)=0$

$\frac{-4 \pm \sqrt{16-4(1)(5)}}{2}$

$\frac{-4 \pm \sqrt{-4}}{2}$

$-2+i, -2-i$

$x = 4, -2+i, -2-i$

No Calculator

Given 2 is a zero of $f(x) = x^3 - 6x^2 + 13x - 10$, find ALL the zeros of the function.

No Calculator

Given 2 is a zero of $f(x) = x^3 - 6x^2 + 13x - 10$, find ALL the zeros of the function.

2	1	-6	13	-10
		2	-8	10
	1	-4	5	0

$$(x-2)(x^2-4x+5)=0$$

$$\frac{4 \pm \sqrt{16-4(1)(5)}}{2}$$

$$\frac{4 \pm \sqrt{-4}}{2}$$

$$2+i, 2-i$$

$$x = 2, 2+i, 2-i$$

No Calculator

Given −3 is a zero of $f(x) = x^3 + 3x^2 + x + 3$, find ALL the zeros of the function.

No Calculator

Given −3 is a zero of $f(x) = x^3 + 3x^2 + x + 3$, find ALL the zeros of the function.

-3	1	3	1	3
		-3	0	-3
	1	0	1	0

$$(x+3)(x^2+1)=0$$

$$x^2 = -1$$

$$x = i, -i$$

$$x = -3, i, -i$$