LESSON 72 – POWER SERIES

Math HL2 - Santowski

OPENING EXERCISE #1

- Expand the following series (6 – 8 terms) and comment upon the similarities and differences you notice:

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{x^n}{n}$$
$$\text{(c)} \sum_{n=1}^{\infty} \frac{2^n}{n!} \qquad \text{(d)} \sum_{n=1}^{\infty} \frac{(2x)^n}{n!}$$
$$\text{(e)} \sum_{n=1}^{\infty} n \qquad \text{(f)} \sum_{n=1}^{\infty} nx^n$$
$$\text{(g)} \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!} \qquad \text{(h)} \sum_{n=1}^{\infty} \frac{10^n x^n}{(n+1)!}$$

OPENING EXERCISE #2

- For the following series, write the first 5 terms and then test the convergences of the following series:
- :

$$\text{(a)} \sum_{k=1}^{\infty} \frac{2^k}{k!} \quad \text{(b)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \quad \text{(c)} \sum_{k=1}^{\infty} \frac{(-4)^k}{k!} \quad \text{(d)} \sum_{k=1}^{\infty} \frac{(0.5)^k}{k!} \quad \text{(e)} \sum_{k=1}^{\infty} \frac{214^k}{k!}$$

OPENING EXERCISE #2

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- Make a general conclusion about the convergence of $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ and prove that it is true:
- The series $\sum_{k=1}^{\infty} \frac{x^k}{k!}$ is an example of a POWER SERIES

OPENING EXERCISE #3

- Use DESMOS to:
- (1) graph $f(x) = \dfrac{1}{1-x}$
- (2) graph $\sum_{n=0}^{\infty} x^n$ so maybe use 100 as the upper limit rather than infinity
- (3) now expand $g(x) = \sum_{n=0}^{\infty} x^n = \ldots\ldots$ and call it $g(x)$
- (4) Evaluate & compare f(0.2) and g(0.2); f(-0.1) and g(-0.1); f(0.01) and g(0.01)

OPENING EXERCISE #3

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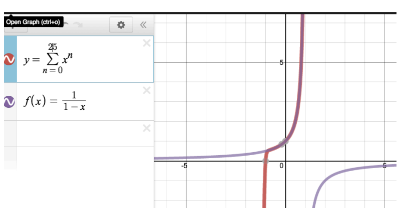
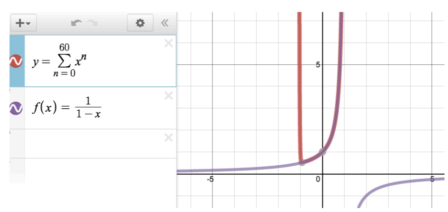


OPENING EXERCISE #3



OPENING EXERCISE #3

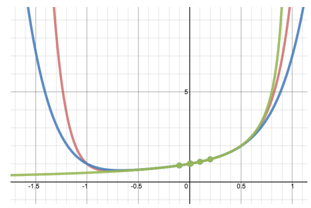
$g(x) = 1 + x + x^2 + x^3 + x^4$

x	$g(x)$
.1	1.111111
0.2	1.249984
−0.1	0.909091
.01	1.010101
2	127

$f(x) = \frac{1}{1-x}\{x<1\}$

x	$f(x)$
0.1	1.1111111
0.2	1.25
−0.1	0.90909091
0.01	1.010101
0	1
1	undefined
2	undefined

OPENING EXERCISE #3



POWER SERIES

- So, what is a power series OR what do we mean by the term "power series"

What are Power Series?

It's convenient to think of a power series as an ***infinite polynomial***:

Polynomials: $2 - x + 3x^2 - 12x^5$

$$1 + (x-1) + 3(x-1)^2 - \left(\tfrac{1}{4}\right)(x-1)^3$$

Power Series: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots = \sum_{k=0}^{\infty}(k+1)x^k$

$$1 - \frac{(x+3)}{3!} + \frac{(x+3)^2}{5!} - \frac{(x+3)^3}{7!} + \frac{(x+3)^4}{9!} - \ldots = \sum_{k=0}^{\infty}\frac{(-1)^k(x+3)^k}{(2k+1)!}$$

OBJECTIVES

- In the next few lessons, we will introduce a few concepts:
  - conditions for convergence of a power series
  - radius of convergence of a power series
  - representation of functions with a power series
- After working through these lessons and completing a sufficient number of exercises on paper, you should be able to
  - determine the radius of convergence of that series
  - expand a function in a power series

A power series is in this form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

or

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

The coefficients $c_0, c_1, c_2 \ldots$ are constants.

The center "a" is also a constant.

(The first series would be centered at the origin if you graphed it. The second series would be shifted left or right. "a" is the new center.)

In general. . .

Definition: A power series is a (family of) series of the form

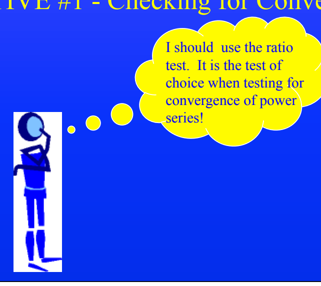
$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

In this case, we say that the power series is based at x_0 or that it is centered at x_0.

What can we say about convergence of power series?

A great deal, actually.

OBJECTIVE #1 - Checking for Convergence



Checking for Convergence

Checking on the convergence of

$$1+2x+3x^2+4x^3+5x^4+\ldots=\sum_{k=0}^{\infty}(k+1)x^k$$

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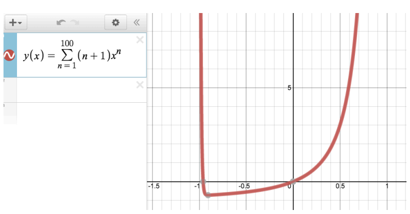
$$1+2x+3x^2+4x^3+5x^4+\ldots=\sum_{k=0}^{\infty}(k+1)x^k$$

We start by setting up the appropriate limit.

$$\lim_{k\to\infty}\frac{(k+2)|x|^{k+1}}{(k+1)|x|^k}=\lim_{k\to\infty}\frac{(k+2)|x|}{(k+1)}=|x|$$

The ratio test says that the series converges provided that this limit is less than 1. That is, when $|x|<1$.

GRAPHIC VERIFICATION



Now you work out the convergence of

$$1-\frac{(x+3)}{3}+\frac{(x+3)^2}{5}-\frac{(x+3)^3}{7}+\frac{(x+3)^4}{9}-\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k(x+3)^k}{(2k+1)}$$

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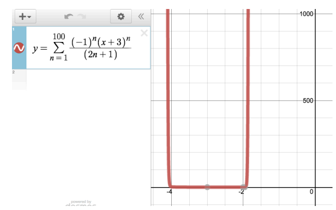
We start by setting up the ratio test limit.

$$\lim_{k\to\infty}\frac{\dfrac{|x+3|^{k+1}}{(2(k+1)+1)}}{\dfrac{|x+3|^{k}}{(2k+1)}}=\lim_{k\to\infty}\frac{|x+3|^{k+1}}{|x+3|^{k}}\frac{(2k+1)}{(2k+3)}=|x+3|\lim_{k\to\infty}\frac{(2k+1)}{(2k+3)}=|x+3|$$

What does this tell us?

- The power series converges absolutely when $|x+3|<1$.
- The power series diverges when $|x+3|>1$.
- The ratio test is inconclusive for x = -4 and x = -2. (Test these separately… what happens?)

GRAPHIC VERIFICATION



What about the convergence of

$$1-\frac{(x+3)}{3!}+\frac{(x+3)^2}{5!}-\frac{(x+3)^3}{7!}+\frac{(x+3)^4}{9!}-\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k(x+3)^k}{(2k+1)!}?$$

What about the convergence of

$$1-\frac{(x+3)}{3!}+\frac{(x+3)^2}{5!}-\frac{(x+3)^3}{7!}+\frac{(x+3)^4}{9!}-\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k(x+3)^k}{(2k+1)!}?$$

We start by setting up the ratio test limit.

$$\lim_{k\to\infty}\frac{\dfrac{|x+3|^{k+1}}{(2(k+1)+1)!}}{\dfrac{|x+3|^{k}}{(2k+1)!}}=\lim_{k\to\infty}\frac{|x+3|^{k+1}}{|x+3|^{k}}\frac{(2k+1)!}{(2k+3)!}$$

$$=\lim_{k\to\infty}\frac{|x+3|(1\cdot2\cdot3\cdot4\cdot\ldots(2k-1)(2k)(2k+1))}{(1\cdot2\cdot3\cdot4\cdot\ldots(2k-1)(2k)(2k+1)(2k+2)(2k+3))}$$

$$=\lim_{k\to\infty}\frac{|x+3|}{(2k+2)(2k+3)}$$

$$=0$$

Since the limit is 0 (which is less than 1), the ratio test says that the series converges absolutely for all x.

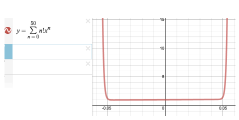
GRAPHIC VERIFICATION

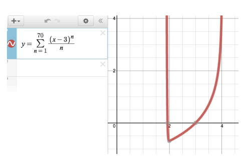
INTERVAL OF CONVERGENCES FROM A GRAPHIC PERSPECTIVE

- Use DESMOS to graph the following power series in order to PREDICT the interval of convergence & the radius of convergence

(a) $\sum_{n=0}^{\infty} n!\,x^n$ (b) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$

GRAPHIC VERIFICATION





Summary - Power Series

Definitions

Interval of Convergence: The interval of x values where the series converges.

Radius of Convergence (R): Half the length of the interval of convergence.

1) The series only converges at $x = a$. $(R = 0)$
2) The series converges for all x values. $(R = \infty)$
3) The series converges for some interval of x.
   $|x - a| < R$
   $(a - R < x < a + R)$

The end values of the interval must be tested for convergence.

The use of the Ratio test is recommended when finding the radius of convergence and the interval of convergence.

Example #1 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❶ $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$

Example #1 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❶ $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$ Ratio Test $c_{n+1} = \frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)(x+3)^{n+1}}{4^{n+1}} \div \frac{(-1)^n n(x+3)^n}{4^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)^n(-1)(n+1)(x+3)^n(x+3)}{4^n(4)} \cdot \frac{4^n}{(-1)^n n(x+3)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right| = \frac{1}{4}|x+3|$$

Example #1 – Power Series

❶ $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$ $\lim_{n\to\infty}\left|\frac{(-1)(n+1)(x+3)}{4n}\right| = \frac{1}{4}|x+3|$

Ratio Test: Convergence for $L < 1$

$\frac{1}{4}|x+3| < 1$

$|x+3| < 4$

Radius of Convergence

$R = 4$

Interval of Convergence:

$-4 < x+3 < 4$

$-7 < x < 1$

End points need to be tested.

Example #1 – Power Series

❶ $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$

$x = -7$

$\sum_{n=0}^{\infty} \frac{(-1)^n n(-7+3)^n}{4^n}$

$\sum_{n=0}^{\infty} \frac{(-1)^n n(-4)^n}{4^n}$

$\sum_{n=0}^{\infty} \frac{(-1)^n n(-1)^n(4)^n}{4^n}$

$\sum_{n=0}^{\infty} n$

n^{th} *term test for diverger*

$\lim_{n\to\infty} n = \infty \neq 0$

divergent at $x = -7$

-7 cannot be included in the interval of convergence

Example #1 – Power Series

❶ $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$

$x = 1$

$\sum_{n=0}^{\infty} \frac{(-1)^n n(1+3)^n}{4^n}$

$\sum_{n=0}^{\infty} \frac{(-1)^n n(4)^n}{4^n}$

$\sum_{n=0}^{\infty} (-1)^n n$

$\sum_{n=0}^{\infty} (-1)^n n$

n^{th} *term test for diverger*

$\lim_{n\to\infty} (-1)^n n = DNE \neq 0$

divergent at $x = 1$

1 cannot be included in the interval of convergence

Therefore, the interval of convergence is:

$-7 < x < 1$

Example #2 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❷ $\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$

Example #2 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❷ $\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$

Ratio Test $c_{n+1} = \frac{2^{n+1}(4x-8)^{n+1}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(4x-8)^{n+1}}{n+1} \div \frac{2^n(4x-8)^n}{n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2^n 2(4x-8)^n(4x-8)}{n+1} \cdot \frac{n}{2^n(4x-8)^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right| = 2|4x-8|$$

Example #2 – Power Series

❷ $\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$ $\quad \lim_{n\to\infty} \left| \frac{2n(4x-8)}{n+1} \right| = 2|4x-8|$

Ratio Test: Convergence for $L < 1$

$2|4x-8| < 1$

$2|4(x-2)| < 1$

$8|x-2| < 1$

$|x-2| < \frac{1}{8}$

Radius of Convergence

$R = \frac{1}{8}$

Interval of Convergence:

$-\frac{1}{8} < x-2 < \frac{1}{8}$

$\frac{15}{8} < x < \frac{17}{8}$

End points need to be tested.

Example #2 - Power Series

❷ $$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$$

$$x = \frac{15}{8}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(4\left(\frac{15}{8}\right)-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{15}{2}-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n(-1)^n}{n \cdot 2^n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Alternating harmonic series is convergent

$\therefore$ *convergent at* $x = \frac{15}{8}$

$\frac{15}{8}$ *can be included in the interval of convergence*

Example #2 - Power Series

❷ $$\sum_{n=1}^{\infty} \frac{2^n(4x-8)^n}{n}$$

$$x = \frac{17}{8}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(4\left(\frac{17}{8}\right)-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{17}{2}-8\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n}$$

$$\sum_{n=1}^{\infty} \frac{2^n}{n \cdot 2^n} \rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$$

Harmonic series is divergent

$\therefore$ *divergent at* $x = \frac{17}{8}$

$\frac{17}{8}$ *cannot be included in the interval of convergence*

Therefore, the interval of convergence is:

$$\frac{15}{8} \leq x < \frac{17}{8}$$

Example #3 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❸ $$\sum_{n=0}^{\infty} n!\,(2x+1)^n$$

Example #3 - Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❸ $$\sum_{n=0}^{\infty} n!\,(2x+1)^n$$ Ratio Test $c_{n+1} = (n+1)!\,(2x+1)^{n+1}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!\,(2x+1)^{n+1}}{n!\,(2x+1)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{n!\,(n+1)(2x+1)^n(2x+1)}{n!\,(2x+1)^n}\right|$$

$$\lim_{n\to\infty}|(n+1)(2x+1)| = \infty > 1$$

The series will converge at one point.

The limit is zero at

$$x = -\frac{1}{2}$$

Radius of Convergence: $R = 0$

The interval of convergence is:

$$x = -\frac{1}{2}$$

Example #4 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❹ $\displaystyle\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}$

Example #4 – Power Series

Example: Find the Radius of Convergence and the Interval of Convergence for the following power series

❹ $\displaystyle\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}$ Ratio Test $c_{n+1} = \dfrac{(x-6)^{n+1}}{n^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{(x+6)^{n+1}}{n^{n+1}} \div \frac{(x+6)^n}{n^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+6)^n(x+6)}{n^n n} \cdot \frac{n^n}{(x+6)^n}\right|$$

$$\lim_{n\to\infty}\left|\frac{(x+6)}{n}\right| = \mathbf{0 < 1}$$

The limit is zero regardless of the value of x.

The series will converge for every x.

Radius of Convergence: $R = \infty$

The interval of convergence is:

$$-\infty < x < \infty$$

- http://www.millersville.edu/~bikenaga/calculus/intervals-of-convergence/intervals-of-convergence.html
- http://blogs.ubc.ca/infiniteseriesmodule/units/unit-3-power-series/power-series/a-motivating-problem-for-power-series/
- https://www.youtube.com/watch?v=Sw7PcBTqE0A
- https://www.youtube.com/watch?v=XWGPjZK0Yzw