Lesson 69 – Ratio Test & Comparison Tests

Math HL – Calculus Option

Series known to converge or diverge

1. A geometric series with | r | <1 converges
2. A repeating decimal converges
3. Telescoping series converge

A necessary condition for convergence:
Limit as *n* goes to infinity for nth term in sequence is 0.

nth term test for divergence:
If the limit as n goes to infinity for the nth term is not 0, the series DIVERGES!

Convergent and Divergent Series

- If the infinite series has a sum, or limit, the series is convergent.
- If the series is not convergent, it is divergent.

Ways To Determine Convergence/ Divergence

- 1. Arithmetic – since no sum exists, it diverges
- 2. Geometric:
  - If |r| > 1, diverges
  - If |r| < 1, converges since the sum exists
- 3. Ratio Test (discussed in a few minutes)

Example

Determine whether each series is convergent or divergent.

- 1/8 + 3/20 + 9/50 + 27/125 + . . .
- 18.75+17.50+16.25+15.00+ . . .
- 65 + 13 + 13/5 + 13/25 . . .

Example

Determine whether each series is convergent or divergent.

- 1/8 + 3/20 + 9/50 + 27/125 + . . .
  - **r=6/5 → |r|>1 → divergent**
- 18.75+17.50+16.25+15.00+ . . .
  - **Arithmetic series → divergent**
- 65 + 13 + 13/5 + 13/25 . . .
  - **r=1/5 → |r|<1 → convergent**

Analytical Tools – Ratio Test

- When a series is neither arithmetic or geometric, it is more difficult to determine whether the series is convergent or divergent.
- So we need a variety of different analytical tools to help us decide whether a series converges or diverges

Ratio Test

In the ratio test, we will use a ratio of a_n and a_{n+1} to determine the convergence or divergence of a series.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, the $\sum a_n$ converges absolutely

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, the $\sum a_n$ diverges.

If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the ratio test is inconclusive.

Ex 1: Test for convergence or divergence of: $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$

Ex 1: Test for convergence or divergence of: $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$

$a_n = n\left(\frac{2}{3}\right)^n \qquad a_{n+1} = (n+1)\left(\frac{2}{3}\right)^{n+1}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n} = \frac{n+1}{n}\left(\frac{2}{3}\right)^{n+1-n} = \frac{n+1}{n}\left(\frac{2}{3}\right)$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}\lim_{n\to\infty}\frac{n+1}{n} = \frac{2}{3}$$

Since this ratio is less than 1, the series converges.

Ex 2: Test for convergence or divergence of: $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Ex 2: Test for convergence or divergence of: $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

$a_n = \frac{n^2}{2^n} \qquad a_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \frac{(n+1)^2}{n^2}\,\frac{2^n}{2^{n+1}} = \frac{1}{2}\cdot\frac{n^2+2n+1}{n^2}$$

The ratio of the leading coefficients is 1

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}\lim_{n\to\infty}\frac{n^2+2n+1}{n^2} = \frac{1}{2}$$

Since this ratio is less than 1, the series converges.

Ex 3: Test for convergence or divergence of: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

Ex 3: Test for convergence or divergence of: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+2)}{n(n+1)}$

$$|a_n| = \frac{(n+2)}{n(n+1)} \qquad |a_{n+1}| = \frac{(n+3)}{(n+1)(n+2)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(n+3)}{(n+1)(n+2)}}{\frac{(n+2)}{n(n+1)}} = \frac{(n+3)}{\cancel{(n+1)}(n+2)} \cdot \frac{n\cancel{(n+1)}}{(n+2)} = \frac{n(n+3)}{(n+2)^2}$$

Coefficient of n² is 1

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n(n+3)}{(n+2)^2} = 1$$

Since this ratio is 1, the test is inconclusive.

Coefficient of n² is 1

Example 4

- Use the ratio test to determine if the series is convergent or divergent.
  1/2 + 2/4 + 3/8 + 4/16 + . . .

Example 4

- Use the ratio test to determine if the series is convergent or divergent.
  1/2 + 2/4 + 3/8 + 4/16 + . . .

$$a_n = \frac{n}{2^n} \quad and \quad a_{n+1} = \frac{n+1}{2^{n+1}}$$

$$r = \lim_{n\to\infty} \frac{n+1/2^{n+1}}{n/2^n} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \bullet \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = 1/2$$

Since r<1, the series is convergent.

Example 5

- Use the ratio test to determine if the series is convergent or divergent.

  1/2 + 2/3 + 3/4 + 4/5 + . . .

Example 5

- Use the ratio test to determine if the series is convergent or divergent.

  1/2 + 2/3 + 3/4 + 4/5 + . . .

$$a_n = \frac{n}{n+1} \quad and \quad a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

$$r = \lim_{n\to\infty} \frac{{n+1}/{n+2}}{{n}/{n+1}} = \lim_{n\to\infty} \frac{n+1}{n+2} \bullet \frac{n+1}{n} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n} = 1$$

Since r=1, the ratio test provides no information.

Example 6

- Use the ratio test to determine if the series is convergent or divergent.

  2 + 3/2 + 4/3 + 5/4 + . . .

Example 6

- Use the ratio test to determine if the series is convergent or divergent.

  2 + 3/2 + 4/3 + 5/4 + . . .

$$a_n = \frac{n+1}{n} \quad and \quad a_{n+1} = \frac{(n+1)+1}{n+1} = \frac{n+2}{n+1}$$

$$r = \lim_{n\to\infty} \frac{{n+2}/{n+1}}{{n+1}/{n}} = \lim_{n\to\infty} \frac{n+2}{n+1} \bullet \frac{n}{n+1} = \lim_{n\to\infty} \frac{n^2+2n}{n^2+2n+1} = 1$$

Since r=1, the ratio test provides no information.

Example 7

- Use the ratio test to determine if the series is convergent or divergent.

  3/4 + 4/16 + 5/64 + 6/256 + . . .

Example 7

- Use the ratio test to determine if the series is convergent or divergent.

  3/4 + 4/16 + 5/64 + 6/256 + . . .

$$a_n = \frac{n+2}{2^{2n}} \quad and \quad a_{n+1} = \frac{(n+2)+1}{2^{2(n+1)}} = \frac{n+3}{2^{2n+2}}$$

$$r = \lim_{n\to\infty} \frac{n+3 \big/ 2^{2n+2}}{n+2 \big/ 2^{2n}} = \lim_{n\to\infty} \frac{n+3}{2^{2n+2}} \bullet \frac{2^{2n}}{n+2} = \lim_{n\to\infty} \frac{n+3}{2^2(n+2)} = \frac{1}{4}$$

Since r<1, the series is convergent.

Example 8

- Use the ratio test to determine if the series is convergent or divergent.

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + ...$$

Example 8

- Use the ratio test to determine if the series is convergent or divergent.

$$1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + ...$$

$$a_n = \frac{1}{n!} \quad and \quad a_{n+1} = \frac{1}{(n+1)!}$$

$$r = \lim_{n\to\infty} \frac{1 \big/ (n+1)!}{1 \big/ n!} = \lim_{n\to\infty} \frac{1}{(n+1)!} \bullet \frac{n!}{1} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

Since r<1, the series is convergent.

Example 9

- Use the ratio test to determine if the series is convergent or divergent.

$$\frac{2}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{8}{3\cdot 4}+\frac{16}{4\cdot 5}+$$

Example 9

- Use the ratio test to determine if the series is convergent or divergent.

$$\frac{2}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{8}{3\cdot 4}+\frac{16}{4\cdot 5}+$$

$$a_n = \frac{2^n}{n(n+1)} \quad and \quad a_{n+1} = \frac{2^{n+1}}{(n+1)(n+2)}$$

$$r = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)(n+2)}{2^n/n(n+1)} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)(n+2)} \bullet \frac{n(n+1)}{2^n} = \lim_{n\to\infty} \frac{2n}{n+2} = 2$$

Since r>1, the series is divergent.

Comparison Test & Limit Comparison Test

HL Math - Santowski

Comparison test:

If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms, then:

(a) If $\sum_{n=1}^{\infty} b_n$ is convergent and $a_n \le b_n$ for all *n*, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} b_n$ is divergent and $a_n \ge b_n$ for all *n*, then $\sum_{n=1}^{\infty} a_n$ diverges.

(smaller than convergent is convergent)
(bigger than divergent is divergent)

Examples: $\sum_{n=2}^{\infty} \frac{3n}{n^2-2} > \sum_{n=2}^{\infty} \frac{3n}{n^2} = 3\sum_{n=2}^{\infty} \frac{1}{n}$ which is a *divergent* harmonic series. Since the original series is larger by comparison, it is divergent.

$\sum_{n=1}^{\infty} \frac{5n}{2n^3+n^2+1} < \sum_{n=1}^{\infty} \frac{5n}{2n^3} = \frac{5}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent *p*-series. Since the original series is smaller by comparison, it is convergent.

Examples

- Use the Comparison Test to determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^3+n+1}$

(c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ (d) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(e) $\sum_{n=1}^{\infty} \frac{5}{n^2-1}$ (f) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Limit Comparison test:

If the the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms, and if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$

where $0 < c < \infty$ then either both series converge or both series diverge.

Useful trick: *To obtain a series for comparison, omit lower order terms in the numerator and the denominator and then simplify.*

Examples: For the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+3}$ compare to $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a convergent *p*-series.

For the series $\sum_{n=1}^{\infty} \frac{\pi^n+\sqrt{n}}{3^n+n^2}$ compare to $\sum_{n=1}^{\infty} \frac{\pi^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{\pi}{3}\right)^n$ which is a divergent geometric series.

Examples

- Use the Limit Comparison Test to determine the convergence or divergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ (b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+n+3}$

(c) $\sum_{n=1}^{\infty} \frac{\pi^n+\sqrt{n}}{3^n+n^2}$ (d) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}+1}$

(e) $\sum_{n=1}^{\infty} \frac{1}{4n^2-n-1}$ (f) $\sum_{n=1}^{\infty} \frac{4n^2-n+5}{n^5+n^4+2n-2}$

(g) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

Limit Comparison Test

If $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N a positive integer)

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ $0 < c < \infty$ then both $\sum a_n$ and $\sum b_n$ converge or both diverge.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum a_n$ converges if $\sum b_n$ converges.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ then $\sum a_n$ diverges if $\sum b_n$ diverges.

Convergence or divergence?

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$