Lesson 67 - Infinite Series – The Basics

Santowski – HL Math Calculus Option

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Objectives

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the *n*th-Term Test for Divergence of an infinite series.

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Review Question: Example 1

Example: Find the sum of $3-1+\frac{1}{3}-\frac{1}{9}+\cdots$

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Example: Find the sum of $3-1+\frac{1}{3}-\frac{1}{9}+\cdots$

$r=-\frac{1}{3}$

$$S=\left(\frac{a_1}{1-r}\right)=\frac{3}{1-\left(-\frac{1}{3}\right)}$$

$$=\frac{3}{1+\frac{1}{3}}=\frac{3}{\frac{4}{3}}=\;3\times\frac{3}{4}=\frac{9}{4}$$

The sum of the series is $\frac{9}{4}$.

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The sum of the terms of an infinite geometric sequence is called a geometric series.

If $|r| < 1$, then the infinite geometric series

$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$

has the sum $S = \sum_{i=0}^{\infty} a_1 r^i = \boxed{\frac{a_1}{1-r}}.$

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Review Question: Example #2

- Ex 1 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

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Review Question: Example 2

- Ex 1 $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

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Partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$

$s_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$

$s_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$

$s_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$

and

$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots \frac{1}{n(n+1)} = \frac{n}{n+1}$

If the sequence of partial sums converges, the series converges

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \ldots \frac{n}{n+1} \ldots$ Converges to 1 so **series** converges.

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Review Question: Example #3

- Determine the values of the following infinite series:

(a) $\sum_{n=1}^{\infty} n$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$

(c) $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$

(d) $\sum_{n=2}^{\infty} \frac{1}{3^{n-1}}$

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Infinite Series

One important application of infinite sequences is in representing "infinite summations."

Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

Infinite series

is an **infinite series** (or simply a **series**).

The numbers a_1, a_2, a_3, are the **terms** of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\Sigma\, a_n$.

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Infinite Series

In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums.**

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n \end{aligned}$$

If this sequence of partial sums converges, the series is said to converge.

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Infinite Series

DEFINITIONS OF CONVERGENT AND DIVERGENT SERIES

For the infinite series $\sum_{n=1}^{\infty} a_n$, the ***n*th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n=1}^{\infty} a_n$ **converges.** The limit S is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots \qquad S = \sum_{n=1}^{\infty} a_n$$

If $\{S_n\}$ diverges, then the series **diverges.**

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Example 1(a) – *Convergent and Divergent Series*

Does the following series converge or diverge?

HOW do we decide?

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

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Example 1(a) – *Convergent and Divergent Series*

The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

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Example 1(a) – *Convergent and Divergent Series* cont'd

Because $\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$

it follows that the series converges and its sum is 1.

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Example 1(b) – *Convergent and Divergent Series* cont'd

Does the following series converge or diverge?

HOW do we decide?

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

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Example 1(b) – *Convergent and Divergent Series*

cont'd

The series $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + 1 + \cdots$

diverges because $S_n = n$ and the sequence of partial sums diverges.

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Example 1(c) – *Convergent and Divergent Series*

cont'd

Does the following series converge or diverge?

HOW do we decide?

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

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Example 1(c) – *Convergent and Divergent Series*

cont'd

The *n*th partial sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of S_n is 1, the series converges and its sum is 1.

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Infinite Series

The series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$$

is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots.$$

Telescoping series

Note that b_2 is canceled by the second term, b_3 is canceled by the third term, and so on.

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Infinite Series

Because the nth partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$.

Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n \to \infty} b_{n+1}.$$

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Geometric Series

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Geometric Series

The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is a **geometric series.**

In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0$$

Geometric series

is a **geometric series** with ratio r.

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Geometric Series

THEOREM 6 CONVERGENCE OF A GEOMETRIC SERIES

A geometric series with ratio r diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

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Example 3(a) – *Convergent and Divergent Geometric Series*

The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$

$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdot \cdot \cdot$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$.
Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1-r} = \frac{3}{1-(1/2)} = 6.$$

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Example 3(b) – *Convergent and Divergent Geometric Series*

cont'd

The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdot \cdot \cdot$$

has a ratio of $r = \frac{3}{2}$.

Because $|r| \geq 1$, the series diverges.

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Examples: Geometric Series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Each term is obtained from the preceding number by multiplying by the same number r.

Find r (the common ratio)

$$\frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \frac{1}{625} + \ldots \qquad \frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \frac{16}{3} + \ldots$$

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Examples: Geometric Series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \qquad \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$$

$$\frac{3}{5} - \frac{12}{25} + \frac{48}{125} - \frac{192}{625} + \ldots \qquad \sum_{n=1}^{\infty} \frac{3}{5}\left(-\frac{4}{5}\right)^{n-1}$$

$$\frac{2}{3} + \frac{4}{3} + \frac{8}{3} + \frac{16}{3} + \ldots \qquad \sum_{n=1}^{\infty} \frac{2}{3}(2)^{n-1}$$

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Repeating decimals

Does the following series converge or diverge?

HOW do we decide?

$$0.080\overline{808} = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \ldots$$

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Repeating decimals Geometric Series

$$0.080\overline{808} = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \ldots$$

$$a = \frac{8}{10^2} \ and \ r = \frac{1}{10^2}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=1}^{\infty} \frac{8}{10^2}\left(\frac{1}{10^2}\right)^{n-1}$$

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} = \frac{\frac{8}{10^2}}{1-\frac{1}{10^2}} = \frac{8}{99}$$

The repeating decimal is equivalent to 8/99.

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Geometric Series

THEOREM 7 PROPERTIES OF INFINITE SERIES

Let Σa_n and Σb_n be convergent series, and let A, B, and c be real numbers. If $\Sigma a_n = A$ and $\Sigma b_n = B$, then the following series converge to the indicated sums.

1. $\sum_{n=1}^{\infty} ca_n = cA$
2. $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$
3. $\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

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Necessary Conditions for Convergence:

*n*th-Term Test for Divergence

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*n*th-Term Test for Divergence

THEOREM 8 LIMIT OF THE *n*TH TERM OF A CONVERGENT SERIES

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

The contra-positive of Theorem 8 provides a useful test for *divergence*. This ***n*th-Term Test for Divergence** states that if the limit of the *n*th term of a series does *not* converge to 0, the series must diverge.

THEOREM 9 *n*TH-TERM TEST FOR DIVERGENCE

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

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Example 5 – *Using the nth-Term Test for Divergence*

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have:

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$, you have:

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Example 5 – *Using the nth-Term Test for Divergence*

a. For the series $\sum_{n=0}^{\infty} 2^n$, you have:

$$\lim_{n\to\infty} 2^n = \infty.$$

So, the limit of the *n*th term is not 0, and the series diverges.

b. For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$, you have:

$$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2}.$$

So, the limit of the *n*th term is not 0, and the series diverges.

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Example 5 – *Using the nth-Term Test for Divergence*

cont'd

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have:

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Example 5 – *Using the nth-Term Test for Divergence*

cont'd

c. For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Because the limit of the *n*th term is 0, the *n*th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence.

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Example

- Does the following series converge or diverge?

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

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