Lesson 66 – Improper Integrals

HL Math - Santowski

Lesson Objectives:

we will learn:

How to solve definite integrals where the interval is infinite and where the function has an infinite discontinuity.

Setting the Stage:

Recall from Lessons 46-50 what definite integrals are …….

TECHNIQUES OF INTEGRATION

In defining a definite integral $\int_a^b f(x)\,dx$, we dealt with a function *f* defined on a finite interval [*a*, *b*] and we assumed that *f* does not have an infinite discontinuity (Lessons 46 – 50).

Setting the Stage

Given the following integrals, PREDICT their value and explain the reasoning behind your prediction.

(a) $\int_1^\infty \frac{1}{x}\,dx$ (b) $\int_1^\infty \frac{1}{x^2}\,dx$ (c) $\int_1^\infty \frac{1}{x^3}\,dx$

Now, evaluate the following integrals (via technology – i.e. www.wolframalpha.com)

Explain ….. ?

Setting the Stage – Challenges with Integrals

Consider the following integrals and explain why evaluating these integrals MIGHT present a bit of an initial problem ……

(a) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ (b) $\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx$ (c) $\int_{-\infty}^{0} xe^x\,dx$

IMPROPER INTEGRALS

In this section, we extend the concept of a definite integral to the cases where:

- The interval is infinite
- *f* has an infinite discontinuity in [*a*, *b*]

In either case, the integral is called an improper integral.

TYPE 1—INFINITE INTERVALS

Consider the infinite region *S* that lies:

- Under the curve $y = 1/x^2$
- Above the *x*-axis
- To the right of the line $x = 1$

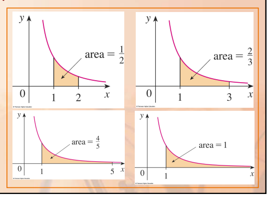
INFINITE INTERVALS

You might think that, since *S* is infinite in extent, its area must be infinite.

- However, let's take a closer look.

INFINITE INTERVALS

The area of the shaded region approaches 1 as $t \to \infty$.

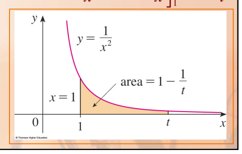


INFINITE INTERVALS

The area of the part of S that lies to the left of the line $x = t$ (shaded) is:

$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}$$

- Notice that $A(t) < 1$ no matter how large t is chosen.



INFINITE INTERVALS

We also observe that:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

INFINITE INTERVALS

So, we say that the area of the infinite region S is equal to 1 and we write:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = 1$$

INFINITE INTERVALS

Using this example as a guide, we define the integral of f (not necessarily a positive function) over an infinite interval as the limit of integrals over finite intervals.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 a**

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 b**

If $\int_t^b f(x)\,dx$ exists for every number $t \leq a$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

CONVERGENT AND DIVERGENT **Definition 1 b**

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

IMPROPER INTEGRAL OF TYPE 1 **Definition 1 c**

If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

- Here, any real number a can be used.

IMPROPER INTEGRALS OF TYPE 1

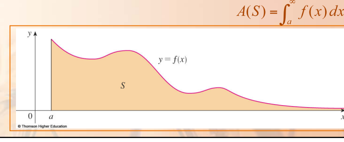
Any of the improper integrals in Definition 1 can be interpreted as an area provided *f* is a positive function.

IMPROPER INTEGRALS OF TYPE 1

For instance, in case (a), suppose $f(x) \geq 0$ and the integral $\int_a^\infty f(x)\,dx$ is convergent.

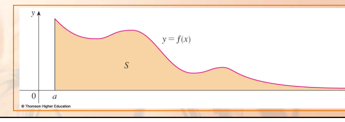
- Then, we define the area of the region $S = \{(x, y) \mid x \geq a,\ 0 \leq y \leq f(x)\}$ in the figure as:

$$A(S) = \int_a^\infty f(x)\,dx$$



IMPROPER INTEGRALS OF TYPE 1

This is appropriate because $\int_a^\infty f(x)\,dx$ is the limit as $t \to \infty$ of the area under the graph of *f* from *a* to *t*.



IMPROPER INTEGRALS OF TYPE 1

Determine whether the integral

$$\int_1^\infty (1/x)\,dx$$

is convergent or divergent.

IMPROPER INTEGRALS OF TYPE 1

According to Definition 1 a, we have:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln|x|\Big]_1^t$$
$$= \lim_{t\to\infty}(\ln t - \ln 1)$$
$$= \lim_{t\to\infty} \ln t = \infty$$

- The limit does not exist as a finite number.
- So, the integral is divergent.

IMPROPER INTEGRALS OF TYPE 1

Let's compare the result of this example with the example at the beginning of the section:

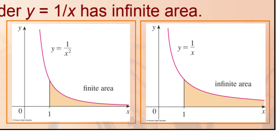
$$\int_1^\infty \frac{1}{x^2}\,dx \text{ converges} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges}$$

- Geometrically, this means the following.

IMPROPER INTEGRALS OF TYPE 1

The curves $y = 1/x^2$ and $y = 1/x$ look very similar for $x > 0$.

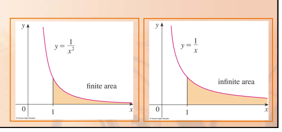
However, the region under $y = 1/x^2$ to the right of $x = 1$ has finite area, but the corresponding region under $y = 1/x$ has infinite area.



IMPROPER INTEGRALS OF TYPE 1

Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$, but $1/x^2$ approaches faster than $1/x$.

- The values of $1/x$ don't decrease fast enough for its integral to have a finite value.



IMPROPER INTEGRALS OF TYPE 1 **Generalization**

For what values of *p* is the integral $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?

- We know from Example 1 that, if $p = 1$, the integral is divergent.
- So, let's assume that $p \neq 1$.

IMPROPER INTEGRALS OF TYPE 1 Generalization

Then,

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty}\int_1^t x^{-p}\,dx$$
$$= \lim_{t\to\infty}\left.\frac{x^{-p+1}}{-p+1}\right]_{x=1}^{x=t}$$
$$= \lim_{t\to\infty}\frac{1}{1-p}\left[\frac{1}{t^{p-1}}-1\right]$$

IMPROPER INTEGRALS OF TYPE 1 Generalization

If $p > 1$, then $p - 1 > 0$.

So, as $t \to \infty$, $t^{p-1} \to \infty$ and $1/t^{p-1} \to 0$.

- Therefore, $\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1}$ if $p > 1$
- So, the integral converges.

IMPROPER INTEGRALS OF TYPE 1 Generalization

However, if $p < 1$, then $p - 1 < 0$.

So, $\frac{1}{t^{p-1}} = t^{1-p} \to \infty$ as $t \to \infty$

- Thus, the integral diverges.

IMPROPER INTEGRALS OF TYPE 1 Definition 2

We summarize the result of Example 4 for future reference:

$\int_1^\infty \frac{1}{x^p}\,dx$ is:

- Convergent if $p > 1$
- Divergent if $p \le 1$

IMPROPER INTEGRALS OF TYPE 1 Example 2

Evaluate $\int_{-\infty}^{0} xe^x\,dx$

- Using Definition 1 b, we have:

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}\int_t^0 xe^x\,dx$$

IMPROPER INTEGRALS OF TYPE 1 Example 2

- We integrate by parts with $u = x$, $dv = e^x\,dx$ so that $du = dx$, $v = e^x$:

$$\int_t^0 xe^x\,dx = \left.xe^x\right]_t^0 - \int_t^0 e^x\,dx$$
$$= -te^t - 1 + e^t$$

IMPROPER INTEGRALS OF TYPE 1 Example 2

- We know that $e^t \to 0$ as $t \to -\infty$, and, by l'Hospital's Rule, we have:

$$\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}}$$
$$= \lim_{t\to-\infty}\frac{1}{-e^{-t}}$$
$$= \lim_{t\to-\infty}(-e^t)$$
$$= 0$$

IMPROPER INTEGRALS OF TYPE 1 Example 2

- Therefore,

$$\int_{-\infty}^{0} xe^x\,dx = \lim_{t\to-\infty}(-te^t - 1 + e^t)$$
$$= -0 - 1 + 0$$
$$= -1$$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}\,dx$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

We need to use integration by parts.

$u=1-x \qquad du=-dx \qquad v=-e^{-x} \qquad dv=e^{-x}dx$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

We need to use integration by parts.

$u=1-x \qquad du=-dx \qquad v=-e^{-x} \qquad dv=e^{-x}dx$

$$\int_0^{+\infty}(1-x)e^{-x}dx=-e^{-x}(1-x)-\int e^{-x}dx$$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

We need to use integration by parts.

$u=1-x \qquad du=-dx \qquad v=-e^{-x} \qquad dv=e^{-x}dx$

$$\int_0^{+\infty}(1-x)e^{-x}dx=-e^{-x}(1-x)-\int e^{-x}dx$$

$$\int_0^{+\infty}(1-x)e^{-x}dx=-e^{-x}+xe^{-x}+e^{-x}=xe^{-x}$$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

$$\int_0^{+\infty}(1-x)e^{-x}dx=\lim_{b\to+\infty}\left[xe^{-x}\right]_0^b$$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

$$\int_0^{+\infty}(1-x)e^{-x}dx=\lim_{b\to+\infty}\left[xe^{-x}\right]_0^b$$

$$\lim_{b\to+\infty}=\frac{b}{e^b}$$

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

$$\lim_{b\to+\infty}=\frac{b}{e^b}$$

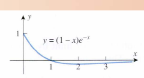
This is of the form $\frac{\infty}{\infty}$ *so we will use L'Hopital's Rule*

Example 3

Evaluate $\int_0^{+\infty}(1-x)e^{-x}dx$

$$\lim_{b\to+\infty}=\frac{b}{e^b}$$

$$\lim_{b\to+\infty}=\frac{1}{e^b}=0$$



The net signed area between the graph and the interval $[0,+\infty)$ is zero.

We can interpret this to mean that the net signed area between the graph of $y=(1-x)e^{-x}$ *and the interval* $[0,+\infty)$ *is 0.*

IMPROPER INTEGRALS OF TYPE 1 **Example 4**

Evaluate $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$

- It's convenient to choose a = 0 in Definition 1 c:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\int_{-\infty}^{0}\frac{1}{1+x^2}dx+\int_{0}^{\infty}\frac{1}{1+x^2}dx$$

IMPROPER INTEGRALS OF TYPE 1 **Example 4**

We must now evaluate the integrals on the right side separately—as follows.

IMPROPER INTEGRALS OF TYPE 1 Example 4

$$\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} \tan^{-1} x \Big]_0^t = \lim_{t\to\infty}(\tan^{-1} t - \tan^{-1} 0) = \lim_{t\to\infty}\tan^{-1} t = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \tan^{-1} x \Big]_t^0 = \lim_{t\to-\infty}(\tan^{-1} 0 - \tan^{-1} t) = 0-\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

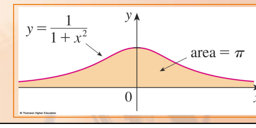
IMPROPER INTEGRALS OF TYPE 1 Example 4

Since both these integrals are convergent, the given integral is convergent and

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

IMPROPER INTEGRALS OF TYPE 1 Example 4

As $1/(1 + x^2) > 0$, the given improper integral can be interpreted as the area of the infinite region that lies under the curve $y = 1/(1 + x^2)$ and above the x–axis.



TYPE 2—DISCONTINUOUS INTEGRANDS

Suppose f is a positive continuous function defined on a finite interval $[a, b)$ but has a vertical asymptote at b.

DISCONTINUOUS INTEGRANDS

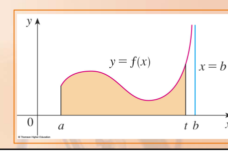
Let S be the unbounded region under the graph of f and above the x-axis between a and b.

- For Type 1 integrals, the regions extended indefinitely in a horizontal direction.
- Here, the region is infinite in a vertical direction.

DISCONTINUOUS INTEGRANDS

The area of the part of S between a and t (shaded region) is:

$$A(t) = \int_a^t f(x)\,dx$$



DISCONTINUOUS INTEGRANDS

If it happens that $A(t)$ approaches a definite number A as $t \to b^-$, then we say that the area of the region S is A and we write:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

DISCONTINUOUS INTEGRANDS

We use the equation to define an improper integral of Type 2 even when f is not a positive function—no matter what type of discontinuity f has at b.

IMPROPER INTEGRAL OF TYPE 2 Definition 3 a

If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

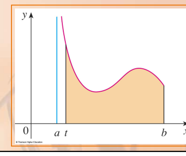
If *f* is continuous on (*a*, *b*] and is discontinuous at *a*, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

Definition 3 b is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at *a* and *c*, respectively.



IMPROPER INTEGRAL OF TYPE 2 **Definition 3 b**

The improper integral $\int_a^b f(x)\,dx$ is called:

- Convergent if the corresponding limit exists.
- Divergent if the limit does not exist.

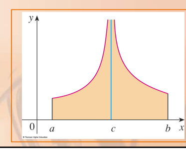
IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

If *f* has a discontinuity at *c*, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

IMPROPER INTEGRAL OF TYPE 2 **Definition 3 c**

Definition 3 c is illustrated for the case where $f(x) \geq 0$ and has vertical asymptotes at *a* and *c*, respectively.



IMPROPER INTEGRALS OF TYPE 2 **Example 5**

Find $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$

- First, we note that the given integral is improper because $f(x) = 1/\sqrt{x-2}$ has the vertical asymptote $x = 2$.

IMPROPER INTEGRALS OF TYPE 2 **Example 5**

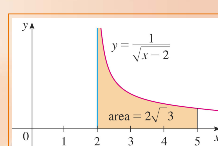
- The infinite discontinuity occurs at the left end-point of [2, 5].
- So, we use Definition 3 b:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2^+}\int_t^5 \frac{dx}{\sqrt{x-2}}\\ &= \lim_{t\to 2^+} 2\sqrt{x-2}\,\Big]_t^5\\ &= \lim_{t\to 2^+} 2(\sqrt{3}-\sqrt{t-2})\\ &= 2\sqrt{3}\end{aligned}$$

- Thus, the given improper integral is convergent.

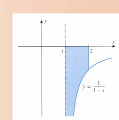
IMPROPER INTEGRALS OF TYPE 2 **Example 5**

- Since the integrand is positive, we can interpret the value of the integral as the area of the shaded region here.



Example 6

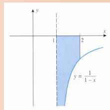
Evaluate $\int_1^2 \frac{dx}{1-x}$



Example 6

Evaluate $\int_1^2 \frac{dx}{1-x}$

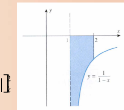
$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\int_k^2 \frac{dx}{1-x}$$



Example 6

Evaluate $\int_1^2 \frac{dx}{1-x}$

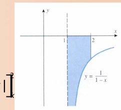
$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2$$



Example 6

Evaluate $\int_1^2 \frac{dx}{1-x}$

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2$$



$$\lim_{k\to 1^+}\Big[-\ln|1-x|\Big]_k^2 = -\ln|-1| + \ln|1-k| = -\infty$$

IMPROPER INTEGRALS OF TYPE 2 Example 7

Determine whether $\int_0^{\pi/2} \sec x\,dx$ converges or diverges.

- Note that the given integral is improper because: $\lim_{x\to(\pi/2)^-} \sec x = \infty$

IMPROPER INTEGRALS OF TYPE 2 Example 7

- Using Definition 2 a and Formula 14 from the Table of Integrals, we have:

$$\int_0^{\pi/2} \sec x\,dx = \lim_{x\to(\pi/2)^-}\int_0^t \sec x\,dx$$
$$= \lim_{x\to(\pi/2)^-} \ln|\sec x + \tan x|\Big]_0^t$$
$$= \lim_{x\to(\pi/2)^-} \Big[\ln(\sec t + \tan t) - \ln 1\Big] = \infty$$

- This is because $\sec t \to \infty$ and $\tan t \to \infty$ as $t \to (\pi/2)^-$.
- Thus, the given improper integral is divergent.

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

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Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\Big[3(x-2)^{1/3}\Big]$$

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\int_1^k \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^-}\Big[3(x-2)^{1/3}\Big]$$

$$\lim_{k\to 2^-}\Big[3(k-2)^{1/3} - 3(1-2)^{1/3}\Big] = 3$$

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+}\left[3(x-2)^{1/3}\right]$$

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}} = \lim_{k\to 2^+}\left[3(x-2)^{1/3}\right]$$

$$\lim_{k\to 2^+}\left[3(4-2)^{1/3} - 3(k-2)^{1/3}\right] = 3\sqrt[3]{2}$$

Example 8

Evaluate $\int_1^4 \frac{dx}{(x-2)^{2/3}} = 3 + 3\sqrt[3]{2}$

IMPROPER INTEGRALS OF TYPE 2 Example 9

Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

- Observe that the line $x = 1$ is a vertical asymptote of the integrand.

IMPROPER INTEGRALS OF TYPE 2 Example 9

- As it occurs in the middle of the interval [0, 3], we must use Definition 3 c with $c = 1$:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\begin{aligned}\int_0^1 \frac{dx}{x-1} &= \lim_{t\to 1^-}\int_0^t \frac{dx}{x-1} = \lim_{t\to 1^-} \ln|x-1|\Big]_0^t \\ &= \lim_{t\to 1^-}(\ln|t-1| - \ln|-1|) \\ &= \lim_{t\to 1^-}\ln(1-t) = -\infty\end{aligned}$$

- This is because $1 - t \to 0^+$ as $t \to 1^-$.

IMPROPER INTEGRALS OF TYPE 2 Example 9

Thus, $\int_0^1 dx/(x-1)$ is divergent.

This implies that $\int_0^3 dx/(x-1)$ is divergent.

- We do not need to evaluate $\int_1^3 dx/(x-1)$.

WARNING

Suppose we had not noticed the asymptote $x = 1$ in Example 9 and had, instead, confused the integral with an ordinary integral.

WARNING

Then, we might have made the following erroneous calculation:

$$\begin{aligned}\int_0^3 \frac{dx}{x-1} &= \ln|x-1|\Big]_0^3 \\ &= \ln 2 - \ln 1 \\ &= \ln 2\end{aligned}$$

- This is wrong because the integral is improper and must be calculated in terms of limits.

WARNING

From now, whenever you meet the symbol $\int_a^b f(x)\,dx$, you must decide, by looking at the function f on $[a, b]$, whether it is either:

- An ordinary definite integral
- An improper integral

IMPROPER INTEGRALS OF TYPE 2 Example 10

Evaluate $\int_0^1 \ln x\,dx$

- We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$.
- Thus, the given integral is improper, and we have:
$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\int_t^1 \ln x\,dx$$

IMPROPER INTEGRALS OF TYPE 2 Example 10

- Now, we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\begin{aligned}\int_t^1 \ln x\,dx &= x\ln x\Big]_t^1 - \int_t^1 dx \\ &= 1\ln 1 - t\ln t - (1-t) \\ &= -t\ln t - 1 + t\end{aligned}$$

IMPROPER INTEGRALS OF TYPE 2 Example 10

- To find the limit of the first term, we use l'Hospital's Rule:

$$\begin{aligned}\lim_{t\to 0^+} t\ln t &= \lim_{t\to 0^+}\frac{\ln t}{1/t} \\ &= \lim_{t\to 0^+}\frac{1/t}{-1/t^2} \\ &= \lim_{t\to 0^+}(-t) \\ &= 0\end{aligned}$$

IMPROPER INTEGRALS OF TYPE 2 Example 10

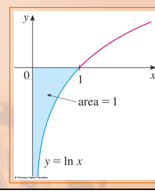
- Therefore,

$$\begin{aligned}\int_0^1 \ln x\,dx &= \lim_{t\to 0^+}(-t\ln t - 1 + t) \\ &= -0 - 1 + 0 \\ &= -1\end{aligned}$$

IMPROPER INTEGRALS OF TYPE 2 Example 10

The geometric interpretation of the result is shown.

- The area of the shaded region above $y = \ln x$ and below the x-axis is 1.



A COMPARISON TEST FOR IMPROPER INTEGRALS

Sometimes, it is impossible to find the exact value of an improper integral and yet it is important to know whether it is convergent or divergent.

- In such cases, the following theorem is useful.
- Although we state it for Type 1 integrals, a similar theorem is true for Type 2 integrals.

COMPARISON THEOREM

Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

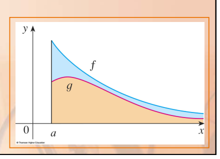
a. If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

b. If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

COMPARISON THEOREM

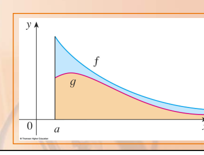
We omit the proof of the theorem.

However, the figure makes it seem plausible.



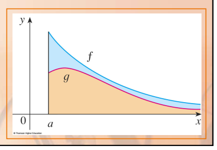
COMPARISON THEOREM

If the area under the top curve $y = f(x)$ is finite, so is the area under the bottom curve $y = g(x)$.



COMPARISON THEOREM

If the area under $y = g(x)$ is infinite, so is the area under $y = f(x)$.



COMPARISON THEOREM

Note that the reverse is not necessarily true:

- If $\int_a^{\infty} g(x)\,dx$ is convergent, $\int_a^{\infty} f(x)\,dx$ may or may not be convergent.
- If $\int_a^{\infty} f(x)\,dx$ is divergent, $\int_a^{\infty} g(x)\,dx$ may or may not be divergent.

COMPARISON THEOREM **Example 11**

Show that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.

- We can't evaluate the integral directly.
- The antiderivative of e^{-x^2} is not an elementary function

COMPARISON THEOREM **Example 11**

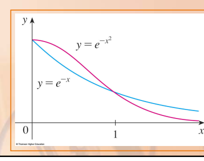
We write:

$$\int_0^{\infty} e^{-x^2}\,dx = \int_0^{1} e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx$$

- We observe that the first integral on the right-hand side is just an ordinary definite integral.

COMPARISON THEOREM **Example 11**

- In the second integral, we use the fact that, for $x \geq 1$, we have $x^2 \geq x$.
- So, $-x^2 \leq -x$ and, therefore, $e^{-x^2} \leq e^{-x}$.



COMPARISON THEOREM **Example 11**

The integral of e^{-x} is easy to evaluate:

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx$$
$$= \lim_{t\to\infty} (e^{-1} - e^{-t})$$
$$= e^{-1}$$

COMPARISON THEOREM **Example 11**

Thus, taking $f(x) = e^{-x}$ and $g(x) = e^{-x^2}$ in the theorem, we see that $\int_1^{\infty} e^{-x^2}\,dx$ is convergent.

- It follows that $\int_0^{\infty} e^{-x^2}\,dx$ is convergent.