Lesson 65 – Infinite Sequences

HL Math - Santowski

Lesson Objectives

- (1) Review basic concepts dealing with sequences
- (2) Evaluate the limits of infinite sequences
- (3) Understand basic concepts associated with limits of sequences
- (4) Introduce limits of functions & make the connection to infinite sequences

Setting the Stage

- Evaluate $\int e^{-x^2} dx$

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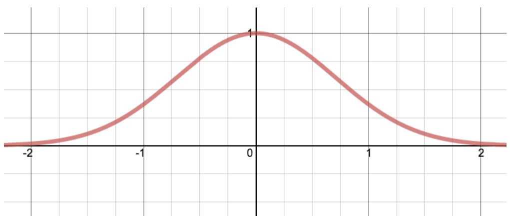
- Explain why we can't evaluate this integral with the techniques discussed so far in this course

Explain the sequence in the following slides:

$f(x) = e^{-x^2}$



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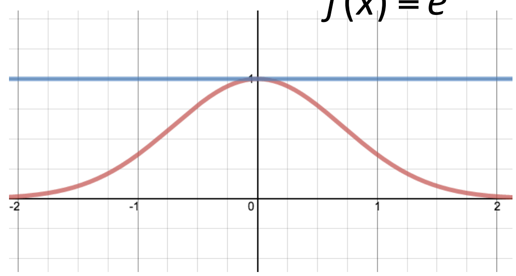

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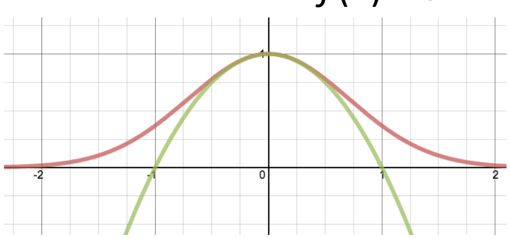

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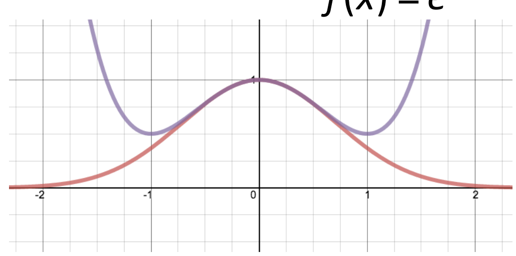

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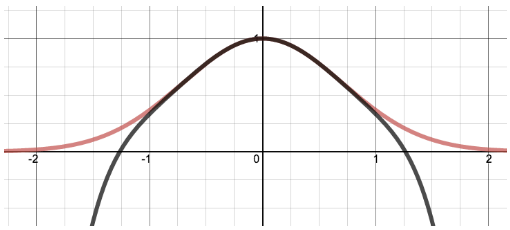

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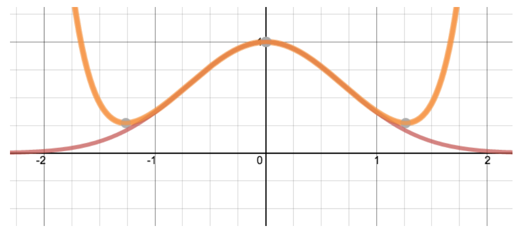

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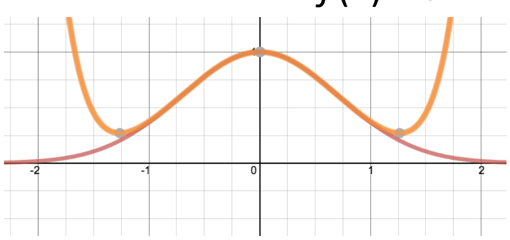


$$f(x) = 1 - \frac{1}{1!}x^2 + \frac{1}{2!}x^4 - \frac{1}{3!}x^6 + \frac{1}{4!}x^8 - \ldots\ldots$$

(A) Review of Sequences

- List the first four terms of each of the following sequences:

(a) $\left\{\frac{2n}{n+1}\right\}_{n=1}^{\infty}$

(b) $\left\{\sqrt{n+4}\right\}_{n=4}^{\infty}$

(c) $\left\{\sin\frac{n\pi}{6}\right\}_{n=1}^{\infty}$

(d) $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$

(A) Review of Sequences

- List the first four terms of each of the following sequences:

(a) $u_{n+1} = 2\left(\sqrt{u_n} + 1\right)$ where $u_1 = 9$

(b) $u_{n+1} = 5 - 2u_n$ where $u_1 = -4$

- Write an explicit expression for the general term of:

$$\left\{\frac{3}{5}, \frac{-4}{25}, \frac{5}{125}, \frac{-6}{625}, \frac{7}{3125}, \ldots\ldots\right\}$$

(B) Limits of a Sequence

- Investigate the behaviour of these sequences:

(a) $u_n = \dfrac{4n^2}{2n^2+5}$

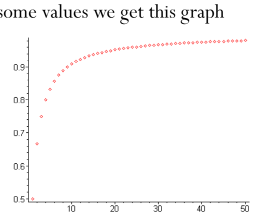
(b) $u_n = \dfrac{4n^2}{2n+5}$

(c) $u_n = \dfrac{(2n-1)!}{(2n+1)!}$

(d) $u_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

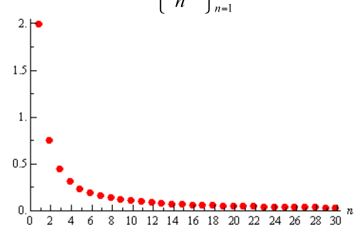
(B) Limit of a sequence

- Consider the sequence $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$
- If we plot some values we get this graph



(B) Limit of a sequence

- Consider the sequence $\left\{\dfrac{n+1}{n^2}\right\}_{n=1}^{\infty}$



(B) Limits of a Sequence

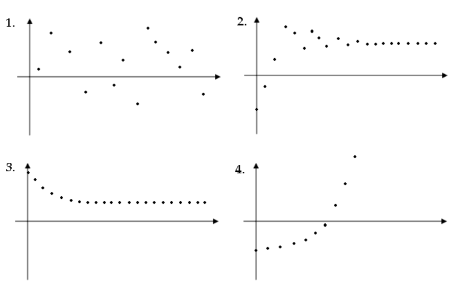
- We say that $\lim_{n\to\infty} a_n = L$ if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.
- We say that $\lim_{n\to\infty} a_n = \infty$ if we can make a_n as large as we want for all sufficiently large n. Again, in other words, the value of the a_n's get larger and larger without bound as n approaches infinity.
- We say that $\lim_{n\to\infty} a_n = -\infty$ if we can make a_n as large and negative as we want for all sufficiently large n. Again, in other words, the value of the a_n's are negative and get larger and larger without bound as n approaches infinity.

(B) Limit of a sequence (Defn 1)

- A sequence $\{a_n\}$ has the limit L if we can make the terms of a_n as close as we like by taking n sufficiently large.
- We write

$$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L \ as \ n \to \infty$$

Limit of a sequence



(B) Limit of a sequence (Defn 2)

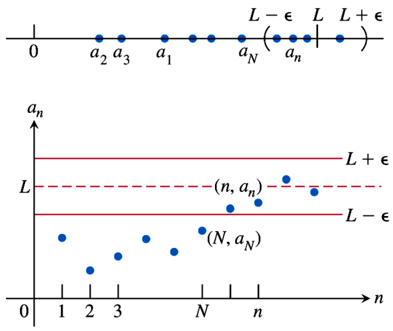
- A sequence $\{a_n\}$ has the limit L if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \varepsilon, \quad whenever \ n > N$$

- We write

$$\lim_{n\to\infty} a_n = L \quad or \quad a_n \to L \ as \ n \to \infty$$

y= L is a horizontal asymptote when sequence converges to L.



(B) Limits of a Sequence

- Example: Given the sequence $u_n = \dfrac{n+1}{2n+1}$
- (a) Find the minimum value of m such that $n \ge m \Rightarrow \left|u_n - \dfrac{1}{2}\right| < 0.1$
- (b) Consider the epsilon value of 0.001 and 0.00001. In each case, find the minimum value of m such that

$$n \ge m \Rightarrow \left|u_n - \frac{1}{2}\right| < \varepsilon$$

Convergence/Divergence

- If $\lim\limits_{n\to\infty} a_n$ exists we say that the sequence **converges**.
  - Note that for the sequence to converge, the limit must be finite
- If the sequence does not converge we will say that it **diverges**
  - Note that a sequence diverges if it approaches to infinity or if the sequence does not approach to anything

(B) Limits of a Sequence

- Which of the following sequences diverge or converge?

(a) $u_n = \dfrac{n^3 + 2n}{n^2 + 4}$

(b) $u_n = \dfrac{3n}{n+4}$

(c) $u_n = \dfrac{(-1)^n}{2^n}$

(d) $u_n = \sin(n)$

Examples

$$\lim_{n\to\infty}\frac{n+4}{n+1} \quad \lim_{n\to\infty}\frac{n-1}{n} \quad \lim_{n\to\infty}\frac{\ln n}{n} \quad \lim_{n\to\infty}\frac{n}{n^2}$$

$$\lim_{n\to\infty}\frac{220n^2}{n^2-4} \quad \lim_{n\to\infty}\frac{n!}{n^n} \quad \lim_{n\to\infty} e^{\frac{3n}{n+1}} \quad \lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$$

$$\lim_{n\to\infty} 5^{\frac{1}{n-2}} \quad \lim_{n\to\infty} 7^{-n} \quad \lim_{n\to\infty}\left(\frac{1}{8}\right)^n \quad \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{2n}$$

$$\lim_{n\to\infty} n^{\frac{1}{n^2}} \quad \lim_{n\to\infty}\frac{\ln n}{n^{\varepsilon}} \quad \lim_{n\to\infty}\frac{n}{e^{\varepsilon n}} \quad \lim_{n\to\infty}\frac{10^n}{n!}$$

(C) More Limit Concepts

- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}\left(a_n \pm b_n\right) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}\left(c\cdot a_n\right) = c\cdot \lim_{n\to\infty} a_n, \quad \lim_{n\to\infty} c = c$$

(C) More Limit Concepts

$$\lim_{n\to\infty}\left(a_n\cdot b_n\right) = \left(\lim_{n\to\infty} a_n\right)\cdot\left(\lim_{n\to\infty} b_n\right)$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty}\left(a_n^{\,p}\right) = \left(\lim_{n\to\infty} a_n\right)^p, \text{ if } p>0 \text{ and } a_n>0$$

L'Hopital and sequences

- L'Hopital: Suppose that f(x) and g(x) are differentiable and that $g'(x)\neq 0$ near a. Also suppose that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Squeeze Theorem for Sequences

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THEOREM 9.3 Squeeze Theorem for Sequences

If

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$$

and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n\to\infty} c_n = L.$$

(C) More Limit Concepts

- Use the Squeeze theorem to investigate the convergence or divergence of:

(a) $u_n = \dfrac{\sin(2n+1)}{n}$

(b) $u_n = \dfrac{\sin(n)}{n^2}$

(c) $\left\{\dfrac{\ln(n)}{n^2}\right\}_{n=1}^{\infty}$

Definition of a Monotonic Sequence

Definition of a Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

or if its terms are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots.$$

Definition of a Bounded Sequence

Definition of a Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number M such that $a_n \le M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number N such that $N \le a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

Bounded Monotonic Sequences

THEOREM 9.5 Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.

Video links patrick jmt

- https://www.youtube.com/watch?v=Kxh7yJC9Jr0
- https://www.youtube.com/watch?v=9K1xx6wfN-U