Lesson 64 L'Hopital's Rule

HL Math - Calculus

Objective: To use L'Hopital's Rule to solve problems involving limits.

Examples to start with

Given our previous work with limits, evaluate the following limits:

$$\lim_{x\to 2}\frac{x^2-4}{x-2} \qquad \lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} \qquad \lim_{x\to 3}\frac{2x-6}{x^2-9}$$

$$\lim_{x\to \infty}\frac{5x-2}{7x+3} \qquad \lim_{x\to \infty}\frac{3x^2+5x-7}{2x^2-3x+1} \qquad \lim_{x\to \infty}\frac{\sqrt{4x^2+1}}{x+3}$$

Challenge example

- So how would we evaluate $\lim_{x\to -2}\frac{x+2}{\ln(x+3)}$

Forms of Type 0/0

- *We looked at several limits in chapter 2 that had the form 0/0. Some examples were:*

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2 \qquad \lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$

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- *The first limit was solved algebraically and the next two were solved using geometric methods. There are other types that cannot be solved by either of these methods. We need to develop a more general method to solve such limits.*

Forms of Type 0/0

- *Lets look at a general example.* $\lim_{x\to a}\frac{f(x)}{g(x)}=0/0$

Forms of Type 0/0

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- *We will assume that f' and g' are continuous at $x = a$ and g' is not zero. Since f and g can be closely approximated by their local linear approximation near a, we can say*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}$$

Forms of Type 0/0

- *Lets look at a general example.* $\lim_{x\to a}\frac{f(x)}{g(x)}=0/0$
- *We know that:*

$$f(a)=\lim_{x\to a}f(x)=0 \qquad g(a)=\lim_{x\to a}g(x)=0$$

- *So we can now say*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}=\frac{f'(a)(x-a)}{g'(a)(x-a)}$$

Forms of Type 0/0

- *Lets look at a general example.* $\lim_{x \to a} \frac{f(x)}{g(x)} = 0/0$
- *This will now become*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)} = \frac{f'(x)}{g'(x)}$$

- *This is know as L'Hopital's Rule.*

Theorem 4.4.1

- *L'Hopital's Rule: Suppose that f and g are differentiable functions on an open interval containing x = a, except possibly at x = a, and that*

$$\lim_{x \to a} f(x) = 0 \quad \textit{and} \quad \lim_{x \to a} g(x) = 0.$$

- *If* $\lim_{x \to a}[f'(x)/g'(x)]$ *exists or if this limit is* $\pm\infty$ *then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example 1

- *Find the following limit.* $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

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- *Find the following limit.* $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
- *Now, we can use L'Hopital's Rule to solve this limit.*

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{2x}{1} = 2(2) = 4$$

Example 2

- *Find the following limit.* $\lim_{x \to 0} \frac{\sin 2x}{x}$

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- *First, we confirm that this of the form 0/0.*
- *Next, we use L'Hopital's Rule to find the limit.*

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \frac{2\cos 2x}{1} = \frac{2(1)}{1} = 2$$

Example 2

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$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0$$

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$$\lim_{x \to 0} \frac{e^x - 1}{x^3} = \frac{e^x}{3x^2} = \frac{1}{0} = +\infty$$

- *This is an asymptote. Do sign analysis to find the answer.* ___+___|___+___

  0

Example 2

- *Find the following limit.* $\lim_{x \to 0^-} \frac{\tan x}{x^2}$

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- *First, we confirm that this of the form 0/0.*
- *Next we use L'Hopital's Rule to find the answer.*

$$\lim_{x \to 0^-} \frac{\tan x}{x^2} = \frac{\sec^2 x}{2x} = \frac{1}{0} = -\infty$$

- *Again, sign analysis.* ___-___|___+___

  0

Example 2

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- *First, we confirm that this of the form 0/0.*
- *Next we use L'Hopital's Rule to find the answer.*

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{\sin x}{2x}=\frac{0}{0}$$

- *We apply L'Hopital's Rule again.*

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$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{\sin x}{2x}=\frac{0}{0}$$

- *We apply L'Hopital's Rule again.*

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{\sin x}{2x}=\frac{\cos x}{2}=\frac{1}{2}$$

Example 2

- *Find the following limit.* $\lim_{x\to +\infty}\frac{x^{-4/3}}{\sin(1/x)}$

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- *First, we confirm that this of the form 0/0.*
- *Next we use L'Hopital's Rule to find the answer.*

$$\lim_{x\to+\infty} \frac{x^{-4/3}}{\sin(1/x)} = \frac{(-4/3)x^{-7/3}}{(-1/x^2)\cos(1/x)} = \frac{(4/3)x^{-1/3}}{\cos(1/x)} = \frac{0}{1} = 0$$

Example 2

- *Find the following limit.* $\lim_{x\to+\infty} \frac{x^{-4/3}}{\sin(1/x)}$

- *First, we confirm that this of the form 0/0.*
- *Next we use L'Hopital's Rule to find the answer.*

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$$\lim_{x\to+\infty} \frac{x^{-4/3}}{\sin(1/x)} = \frac{(-4/3)x^{-7/3}}{(-1/x^2)\cos(1/x)} = \frac{4}{3x^{1/3}\cos(1/x)} = \frac{4}{+\infty} = 0$$

Forms of Type ∞ / ∞

- Theorem 4.4.2: *Suppose that f and g are differentiable functions on an open interval containing x = a, except possibly at x = a, and that*

$$\lim_{x\to a} f(x) = \infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \infty$$

- *If* $\lim_{x\to a}[f'(x)/g'(x)]$ *exists or if this limit is* $\pm\infty$ *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 3

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- *Find the following limits.* $\lim_{x\to+\infty} \frac{x}{e^x}$
- *First, we confirm the form of the equation.*
- *Next, apply L'Hopital's Rule.*

$$\lim_{x\to+\infty} \frac{x}{e^x} = \frac{1}{e^x} = \frac{1}{+\infty} = 0$$

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- *Find the following limits.* $\lim_{x\to 0^+} \frac{\ln x}{\csc x}$
- *First, we confirm the form of the equation.*
- *Next, apply L'Hopital's Rule.* $\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \frac{1/x}{-\csc x \cot x}$
- *This is the same form as the original, and repeated applications of L'Hopital's Rule will yield powers of 1/x in the numerator and expressions involving cscx and cotx in the denominator being of no help. We must try another method.*

Example 3

- *Find the following limits.* $\lim_{x\to 0^+} \frac{\ln x}{\csc x}$
- *We will rewrite the last equation as*

$$\lim_{x\to 0^+} \frac{\ln x}{\csc x} = \frac{1/x}{-\csc x \cot x} = \left(-\frac{\sin x}{x} \tan x\right)$$

$$-\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \tan x = -(1)(0) = 0$$

Examine Exponential Growth

- *Look at the following graphs. What these graphs show us is that e^x grows more rapidly than any integer power of x.*

$$\lim_{x\to+\infty}\frac{x^n}{e^x}=0 \qquad \lim_{x\to+\infty}\frac{e^x}{x^n}=+\infty$$

Forms of Type $0\cdot\infty$

- *There are several other types of limits that we will encounter. When we look at the form $0\cdot\infty$ we can make it look like 0/0 or ∞/∞ and use L'Hopital's Rule.*

Example 4

- *Evaluate the following limit.* $\lim_{x\to0^+} x\ln x$

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- *As written, this is of the form* $\lim_{x\to 0^+} x\ln x = 0\cdot -\infty$
- *Lets rewrite this equation as* $\lim_{x\to 0^+} \frac{\ln x}{1/x} = \frac{-\infty}{+\infty}$
- *Applying L'Hopital's Rule* $\lim_{x\to 0^+} \frac{\ln x}{1/x} = \frac{1/x}{-1/x^2} = -x = 0$

Example 4

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- *Evaluate the following limit.* $\lim_{x\to\pi/4}(1-\tan x)\sec 2x$
- *As written, this is of the form* $\lim_{x\to\pi/4}(1-\tan x)\sec 2x = 0\cdot\infty$
- *Lets rewrite this equation as* $\lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$

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- *As written, this is of the form* $\lim_{x\to\pi/4}(1-\tan x)\sec 2x = 0\cdot\infty$
- *Lets rewrite this equation as* $\lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}$
- *Applying L'Hopital's Rule* $\lim_{x\to\pi/4}\frac{1-\tan x}{\cos 2x}=\frac{-\sec^2 x}{-2\sin 2x}=\frac{-2}{-2}=1$

Forms of Type $\infty \pm \infty$

- *There is another form we will look at. This is a limit of the form* $\infty\pm\infty$*. We will again try to manipulate this equation to match a form we like.*

Example 5

- *Find the following limit.* $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

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- *Lets rewrite the equation as* $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\left(\frac{\sin x - x}{x\sin x}\right)$

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- *Find the following limit.* $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$
- *This is of the form* $\infty - \infty$
- *Lets rewrite the equation as* $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\left(\frac{\sin x - x}{x\sin x}\right)$
- *This is of the form 0/0, so we will use L'Hopital's Rule.*

$$\lim_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)=\frac{\cos x - 1}{x\cos x+\sin x}=\frac{0}{0}=\frac{-\sin x}{-x\sin x+2\cos x}=\frac{0}{2}=0$$

FORMS OF TYPE $0^0, \infty^0, 1^\infty$

- *Limits of the form* $\lim f(x)^{g(x)}$ *give rise to other indeterminate forms of the types:*

$$0^0, \infty^0, 1^\infty$$

Forms of Type $0^0, \infty^0, 1^\infty$

- *Limits of the form* $\lim f(x)^{g(x)}$ *give rise to other indeterminate forms of the types:*

$$0^0, \infty^0, 1^\infty$$

- *Equations of this type can sometimes be evaluated by first introducing a dependent variable* $y = f(x)^{g(x)}$ *and then finding the limit of lny.*

Example 6

- *Show that* $\lim_{x\to 0}(1+x)^{1/x} = e$

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$$y = (1+x)^{1/x} \quad\quad \ln y = (1/x)\ln(1+x)$$

$$\downarrow \quad\quad\quad\quad \downarrow$$

$$\ln y = \ln(1+x)^{1/x} \quad\quad \ln y = \frac{\ln(1+x)}{x}$$

Example 6

- *Show that* $\lim_{x\to 0}(1+x)^{1/x}=e$

$$\ln y=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{0}{0}$$

Example 6

- *Show that* $\lim_{x\to 0}(1+x)^{1/x}=e$

$$\ln y=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{0}{0}$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{1/(1+x)}{1}=\frac{1}{1+x}=1$$

Example 6

- *Show that* $\lim_{x\to 0}(1+x)^{1/x}=e$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\frac{1/(1+x)}{1}=\frac{1}{1+x}=1$$

$$\lim_{x\to 0}\ln y=1$$

- *This implies that* $y\to e$ *as* $x\to 0$.

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Indeterminate Forms

$$\frac{0}{0}\ \ and\ \ \frac{\infty}{\infty}$$

$$\lim_{x\to 0}\frac{2x^3-3x^2+5x}{x^3-x}=\frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{secx}{1+tanx}=\frac{\infty}{\infty}$$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Indeterminate Forms

$\frac{0}{0}$ *and* $\frac{\infty}{\infty}$

$\lim_{x \to 0} \frac{2x^3 - 3x^2 + 5x}{x^3 - x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{6x^2 - 6x + 5}{3x^2 - 1} =$

$\frac{5}{-1} = -5$

$\lim_{x \to \frac{\pi}{2}} \frac{secx}{1 + tanx} = \frac{\infty}{\infty}$

$\lim_{x \to \frac{\pi}{2}} \frac{secxtanx}{(secx)^2} =$

$\lim_{x \to \frac{\pi}{2}} \frac{tanx}{secx} =$

$\lim_{x \to \frac{\pi}{2}} sinx =$

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Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Indeterminate Forms

$\frac{0}{0}$ *and* $\frac{\infty}{\infty}$

$\lim_{x \to 0} \frac{sinx}{x} = \frac{0}{0}$

$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Indeterminate Forms

$\frac{0}{0}$ *and* $\frac{\infty}{\infty}$

$\lim_{x \to 0} \frac{sinx}{x} = \frac{0}{0}$

$\lim_{x \to 0} \frac{cosx}{1} =$

$\frac{1}{1} = 1$

$\lim_{x \to \infty} \frac{e^x}{x^2} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{e^x}{2} =$

$\frac{\infty}{2} = \infty$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms

$0 \cdot \infty$ *and* $\infty - \infty$

$\lim_{x \to \frac{\pi}{2}} (tanx \ln(sinx)) = \ \infty \cdot 0$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms

$0 \cdot \infty$ *and* $\infty - \infty$

$$\lim_{x\to\frac{\pi}{2}}(tanx\ln(sinx)) = \infty \cdot 0$$

$$\lim_{x\to\frac{\pi}{2}}\frac{sinxln(sinx)}{cosx} = \frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{2}}\frac{cosxln(sinx) + sinx\,\frac{1}{sinx}\,cosx}{-sinx} =$$

$$\lim_{x\to\frac{\pi}{2}}\frac{cosxln(sinx) + cosx}{-sinx} = \frac{0}{-1} = 0$$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms

$0 \cdot \infty$ *and* $\infty - \infty$

$$\lim_{x\to 0}\frac{1}{sinx} - \frac{1}{x} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms

$0 \cdot \infty$ *and* $\infty - \infty$

$$\lim_{x\to 0}\frac{1}{sinx} - \frac{1}{x} = \frac{1}{0} - \frac{1}{0} = \infty - \infty$$

Find a common denominator

$xsinx$

$$\lim_{x\to 0}\frac{x}{xsinx} - \frac{sinx}{xsinx} = \lim_{x\to 0}\frac{x - sinx}{xsinx} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{1 - cosx}{sinx + xcosx} = \frac{0}{0}$$

$$\lim_{x\to 0}\frac{sinx}{cosx + cosx - xsinx} = \frac{0}{2} = 0$$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms – Indeterminate Powers

$0^0, \infty^0, 1^\infty$

$$\lim_{x\to 0^+} x^x = 0^0$$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms – Indeterminate Powers

$0^0, \infty^0, 1^\infty$

$\lim_{x \to 0^+} x^x = 0^0$

$y = \lim_{x \to 0^+} x^x$

$lny = ln \lim_{x \to 0^+} x^x$

$lny = \lim_{x \to 0^+} ln\, x^x$

$lny = \lim_{x \to 0^+} xln\, x = \quad 0 \cdot (-\infty)$

$lny = \lim_{x \to 0^+} \frac{lnx}{\frac{1}{x}} = \frac{-\infty}{\infty}$

$lny = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} =$

$lny = \lim_{x \to 0^+} -x = \quad 0$

$lny = 0$

$e^{lny} = e^0$

$y = 1$

$y = \lim_{x \to 0^+} x^x = 1$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms – Indeterminate Powers: $0^0, \infty^0, 1^\infty$

$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = \quad 1^\infty$

Sec 4.5 – Indeterminate Forms and L'Hopital's Rule

Other Indeterminate Forms – Indeterminate Powers

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$lny = \lim_{x \to 0^+} ln\, x^x$

$lny = \lim_{x \to 0^+} xln\, x = \quad 0 \cdot (-\infty)$

$lny = \lim_{x \to 0^+} \frac{lnx}{\frac{1}{x}} = \frac{-\infty}{\infty}$

$lny = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} =$

$lny = \lim_{x \to 0^+} -x = \quad 0$

$lny = 0$

$e^{lny} = e^0$

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$y = \lim_{x \to 0^+} x^x = 1$