Lesson 63 - Intermediate Value Theorem

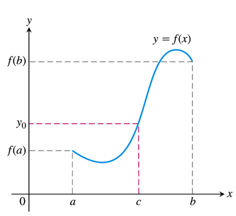
HL Math – Calculus

Intermediate Value Theorem

- Intermediate Value Theorem. Let f be a function which is continuous on the closed interval [a, b]. Suppose that d is a real number between f(a) and f(b); then there exists c in [a, b] such that f(c) = d.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Examples - Context

- If between 7am and 2pm the temperature went from 55 to 70.
  - At some time it reached 62.
  - Time is continuous
- If between his 14th and 15th birthday, a boy went from 150 to 165 lbs.
  - At some point he weighed 155lbs.
  - It may have occurred more than once.

Examples

- (1) Show that a "c" exists such that f(c)=2 for f(c)=x^2 +2x-3 in the interval [0, 2]
- (2) Determine if f(x) has any real roots, where

$$f(x) = x^2 - \sqrt{x+1} \quad \text{in interval}\,[1,2]$$

Example 1

- Show that a "c" exists such that f(c)=2 for f(c)=x^2 +2x-3 in the interval [0, 2]
- First, f(x) is continuous
- Second f(0) = -3
- Third, f(2) = 5
- Thus, by the IVT, there exists a value, x = c, such that f(c) = 2 since -3 < 2 < 5

Example 2

- Determine if f(x) has any real roots, where

$$f(x) = x^2 - \sqrt{x+1} \quad \text{in interval}\,[1,2]$$

- (1) f(x) is continuous on the interval [1,2]
- (2) f(1) = -ve #
- (3) f(2) = +ve #
- $\therefore \text{by IVT} \;\; \exists\; c \in [1,2] \ni \; f(c) = 0$

Examples

- (3) Is any real number exactly one less than its cube?
- (4) Why does the IVT fail to hold for f(x) on [-1, 1]?

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1 \end{cases}$$

Examples

- (5) Show why a root exists in the given interval

$$f(x) = 2x^3 + x^2 + 2 \quad on \quad [-2,-1]$$

- (6) Show why a root exists in the given interval

$$\frac{1}{x} = x^4 + 1 \quad on \quad \left[\tfrac{1}{2}, 1\right]$$

Is any real number exactly one less than its cube?

(Note that this doesn't ask what the number is, only if it exists.)

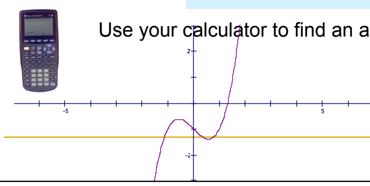
$x = x^3 - 1$ $\quad f(0) = -1$ $\quad f(1) = -1$ $\quad f(2) = 5$

$0 = x^3 - x - 1$

$f(x) = x^3 - x - 1$

Since f is a continuous function, by the intermediate value theorem it must take on every value between -1 and 5. Therefore there must be at least one solution between 1 and 2.

Use your calculator to find an approximate solution.



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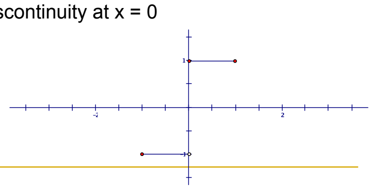
(4) Why does the IVT fail to hold for f(x) on [-1, 1]?

$$f(x) = \begin{cases} -1 & -1 \le x < 0 \\ 1 & 0 \le x \le 1 \end{cases}$$

Not Continuous in interval!

Point of discontinuity at x = 0



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(5) Show why a root exists in the given interval

$$f(x) = 2x^3 + x^2 + 2 \quad on \quad [-2,-1]$$

Continuous in interval

f(-2)= -10

f(-1)= 1

$$\therefore \text{by IVT} \;\; \exists \; c \in [-2,-1] \ni f(c) = 0$$

(6) Show why a root exists in the given interval

$$\frac{1}{x} = x^4 + 1 \quad on \quad \left[\tfrac{1}{2}, 1\right]$$

Continuous in interval

$$f(1/2) = \frac{1}{16} + 1 - 2 = \frac{-15}{16}$$

f(1)= 1

$$\therefore \text{by IVT} \quad \exists \ c \in \left[\frac{1}{2}, 1\right] \ni f(c) = 0$$

Example 2

- Consider the equation sin x = x – 2 . Use the intermediate Value Theorem to explain why there must be a solution between π/2 and π.

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Example 3

- Consider the function ,

$$f(x) = \frac{3}{x+5}$$

- Calculate f(6), f(-5.5), f(0)
- Can you conclude that there must be a zero between f(6) and f(-5.5)?