Lesson 58 – Homogeneous Differential Equations

IBHL - Calculus - Santowski

Lesson Objectives

- Review the previous two types of FODEs that we already know how to solve
- Introduce homogeneous DEs and solve using substitution

(A) Review

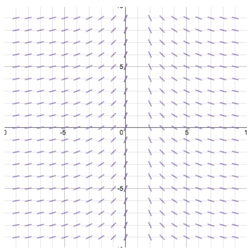
- We have seen two simple types of first order Differential Equations so far in this course and have seen simple methods for solving them algebraically:
- (1) Simple DEs in the form of $\frac{dy}{dx} = f(x)$ wherein we use a simple antiderivative (or integral) to solve the DE
- (2) DEs in the form of $\frac{dy}{dx} = f(x) \times g(y)$ OR $\frac{dy}{dx} = \frac{f(x)}{g(y)}$ wherein we separate the variables in order to solve the DE

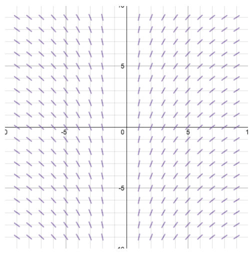
(A) Review - Practice

- Determine the general solution of the following DEs

$$(a)\ \frac{dy}{dx} = \frac{2}{1-x}$$
$$(b)\ \frac{dy}{dx} = \frac{6x+3}{x^2+x}$$
$$(c)\ \frac{dy}{dx} = \sin^2 x$$
$$(d)\ \frac{dy}{dx} = \frac{x\sin x}{e^y}$$
$$(e)\ x\sqrt{1-y^2}\frac{dx}{dy} = 1$$

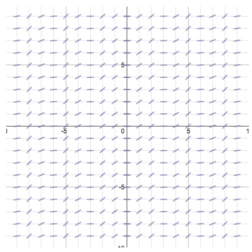
(A) Review – Introducing Slope Fields

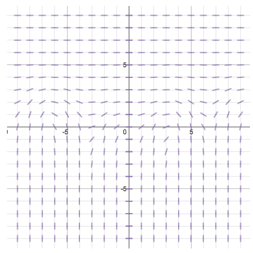




$$\frac{dy}{dx} = \frac{2}{1-x} \qquad \frac{dy}{dx} = \frac{6x+3}{x^2+x}$$

(A) Review – Introducing Slope Fields





$$\frac{dy}{dx} = \sin^2(x) \qquad \frac{dy}{dx} = \frac{x\sin x}{e^y}$$

Homogeneous Functions

A function f (x, y) in x and y is called a homogenous function, if the degrees of each term are equal.

Examples:

$g(x, y) = x^2 - xy + y^2$ is a homogeneous function of degree 2

$f(x, y) = x^3 + 3x^2y + 2y^2x$ is a homogeneous function of degree 3

Homogeneous Differential Equations

$$\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$$

where f (x, y) and g(x, y) is a homogeneous functions of the same degree in x and y, then it is called homogeneous differential equation.

Example: $\frac{dy}{dx} = \frac{y^3 + 3xy^2}{x^3}$ is a homogeneous differential equation as

$y^3 + 3xy^2$ and x^3 both are homogeneous functions of degree 3.

(B) Homogeneous DEs

- A FODE in the form of $\frac{dy}{dx} = f(x,y)$ is homogeneous if it does not depend on x and y separately, but only the ratio of y/x. Homogeneous DEs are written in the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$
- ALGEBRAIC STRATEGY ➔ using a substitution, these DEs can be turned into separable DEs ➔ our substitution will be

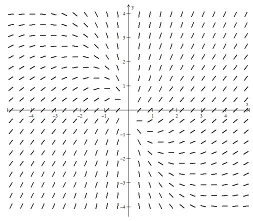
$$v = \frac{y}{x} \quad \text{OR} \quad y = vx$$

(C) Example #1

- Let's work with the DE

$$\frac{dy}{dx} = \frac{x+y}{x}$$

- But first, let's get a visual/graphic perspective from this SLOPE FIELD diagram



(C) Example #1

- Let's work with the DE $\frac{dy}{dx} = \frac{x+y}{x}$
- We will rearrange it (if possible) to a form of y/x
- $$\frac{dy}{dx} = \frac{x+y}{x}$$
$$\frac{dy}{dx} = \frac{x}{x} + \frac{y}{x}$$
$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

(C) Example #1

- Now make the substitution wherein $y = vx$

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x} \quad \Rightarrow \text{let } y = vx$$
$$\frac{d(vx)}{dx} = 1 + \frac{vx}{x}$$
$$\frac{d}{dx}(vx) = 1 + v \quad \Rightarrow \text{ use product rule}$$
$$v\frac{dx}{dx} + x\frac{dv}{dx} = 1 + v \quad \Rightarrow \text{ rearrange}$$
$$v + x\frac{dv}{dx} = 1 + v$$
$$x\frac{dv}{dx} = 1$$
$$\frac{dv}{dx} = \frac{1}{x} \quad \Rightarrow \text{ simple integral or separate}$$

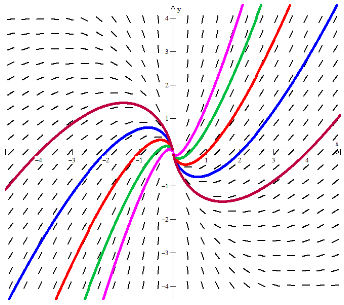
(C) Example #1

- Now we simply integrate and simplify …..

$$\frac{dv}{dx} = \frac{1}{x} \Rightarrow \text{simple integral or separate}$$
$$\therefore v = \ln|x| + C \Rightarrow \text{but recall what v equals?}$$
$$\therefore \frac{y}{x} = \ln|x| + C \Rightarrow \text{replace C with lnC}$$
$$\therefore \frac{y}{x} = \ln|Cx|$$
$$\therefore y = x\ln|Cx|$$

(C) Example #1 – Graphic Solns

$$\frac{dy}{dx} = \frac{x+y}{x}$$

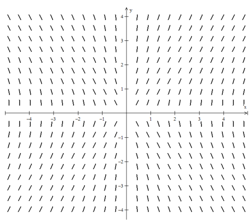


(D) Example #2

- Let's work with the DE

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

- But first, let's get a visual/ graphic perspective from this SLOPE FIELD diagram



(D) Example #2

- Let's work with the DE $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$
- We will rearrange it (if possible) to a form of y/x
- $$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$
$$\frac{dy}{dx} = \frac{x^2}{xy} + \frac{y^2}{xy}$$
$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} = \frac{1}{\left(\frac{y}{x}\right)} + \frac{y}{x}$$

(D) Example #2

- Now make the substitution wherein $y = vx$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{y}{x}\right)} + \frac{y}{x} \quad \Rightarrow \text{now let } y = vx$$

$$\frac{d(vx)}{dx} = \frac{1}{\left(\frac{vx}{x}\right)} + \frac{vx}{x}$$

$$\text{now use product rule} \Rightarrow \frac{d}{dx}(vx) = \frac{1}{v} + v$$

$$v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$v + x\frac{dv}{dx} = \frac{1}{v} + v$$

$$x\frac{dv}{dx} = \frac{1}{v} \Rightarrow \text{rearrange \& separate}$$

$$vdv = \frac{1}{x}dx$$

(D) Example #2

- Now we simply integrate and simplify .....

$$vdv = \frac{1}{x}\text{dx} \Rightarrow \text{simply integrate}$$

$$\therefore \frac{1}{2}v^2 = \ln|x| + C \Rightarrow \text{replace C with lnC}$$

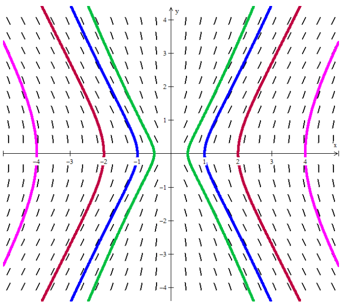
$$\therefore \frac{1}{2}v^2 = \ln|x| + \ln C \Rightarrow \text{but recall what v equals?}$$

$$\therefore \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln|Cx|$$

$$\therefore y^2 = 2x^2\ln|Cx|$$

(D) Example #2 – Graphic Solns

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

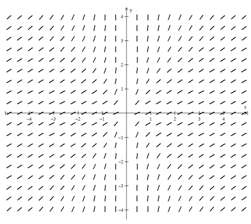


(E) Example #3

- Let's work with the DE

$$2x^2\frac{dy}{dx} = x^2 + y^2$$

- But first, let's get a visual/graphic perspective from this SLOPE FIELD diagram



(E) Example #3

- Let's work with the DE $2x^2\frac{dy}{dx} = x^2 + y^2$
- We will rearrange it (if possible) to a form of y/x

$$2x^2\frac{dy}{dx} = x^2 + y^2$$
$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$$
$$\frac{dy}{dx} = \frac{x^2}{2x^2} + \frac{y^2}{2x^2}$$
$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2 = \frac{1}{2}\left(1 + \left(\frac{y}{x}\right)^2\right)$$

(E) Example #3

- Now make the substitution wherein **v** $= y/x$
-

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + \left(\frac{y}{x}\right)^2\right) \Rightarrow \text{now let } v = {}^{y}\!/\!_{x}$$
$$\frac{d(vx)}{dx} = \frac{1}{2}(1 + v^2)$$
$$\text{now use product rule} \Rightarrow \frac{d}{dx}(vx) = \frac{1}{v} + v$$
$$v\frac{dx}{dx} + x\frac{dv}{dx} = \frac{1}{2}(1 + v^2)$$
$$v + x\frac{dv}{dx} = \frac{1}{2}(1 + v^2)$$
$$2v + 2x\frac{dv}{dx} = 1 + v^2 \quad \Rightarrow \text{rearrange}$$
$$2x\frac{dv}{dx} = 1 - 2v + v^2 \quad \Rightarrow \text{separate}$$
$$\frac{2dv}{(1-v)^2} = \frac{1}{x}dx$$

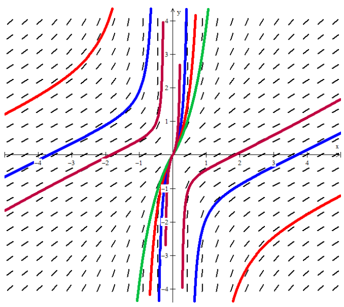
(E) Example #3

- Now we simply integrate and simplify .....

$$\frac{2\text{dv}}{(1\text{-v})^2} = \frac{1}{\text{x}}\text{dx} \Rightarrow \text{simply integrate}$$
$$\therefore \frac{2}{1-v} = \ln|x| + \ln C = \ln|Cx|$$
$$\therefore \frac{2}{\ln|Cx|} = 1 - v$$
$$\therefore v = \frac{y}{x} = 1 - \frac{2}{\ln|Cx|}$$
$$\therefore y = x - \frac{2x}{\ln|Cx|}$$

(E) Example #3 – Graphic Solns

$$2x^2\frac{dy}{dx} = x^2 + y^2$$



(F) Practice Problems

$$(a)\ xy^2\frac{dy}{dx}=x^3+y^3$$

$$(b)\ x^2\frac{dy}{dx}=y^2+2xy$$

$$(c)\ 2xy\frac{dy}{dx}=x^2+y^2$$

$$(d)\ \frac{dy}{dx}=\frac{y(x-y)}{x^2}$$

$$(e)\ \frac{dy}{dx}=\frac{x-y}{x+y}$$

$$(f)\ \frac{dy}{dx}=\frac{y(x^2+y^2)}{xy^2-2x^3}$$

$$(g)\ \left(x+ye^{y/x}\right)dx-\left(xe^{y/x}\right)dy=0$$

(G) Video Resources

- From patrickJMT:

- https://www.youtube.com/watch?v=vEtEAYi2cIA
- https://www.youtube.com/watch?v=-in3FyX6rtM
- https://www.youtube.com/watch?v=QOhjUwiQlG4

- From Mathispower4u
- https://www.youtube.com/watch?v=V_rKXsUIils