Lesson 57 - Applications of DE

Calculus - Santowski

Lesson Objectives

- 1. Apply differential equations to applications involving exponential growth & decay, Newton's Law of Cooling, and introduce the Logistic Equation for population modeling

Fast Five

- Solve the following DE:

1. $x^2y' = y - xy$ and $y(1) = 1$
2. $y' = 6e^{2x-y}$ and $y(0) = 0$
3. $\cos y\,dx + (1 + e^{-x})\sin y\,dy = 0$

(A) Exponential Growth

- Write a DE for the statement: the rate of growth of a population is directly proportional to the population

$$\frac{dP}{dt} \propto P \quad \text{or} \quad \frac{dP}{dt} = kP$$

- Solve this DE:

$$\frac{dP}{dt} = kP$$
$$\int \frac{dP}{P} = \int kdt$$
$$\ln|P| = kt + C$$
$$P(t) = e^{kt+C} = e^C e^{kt}$$
$$P(t) = Ce^{kt}$$

(A) Exponential Growth

- The population of bacteria grown in a culture follows the Law of Natural Growth with a growth rate of 15% per hour. There are 10,000 bacteria after the first hour.
- (a) Write an equation for P(t)
- (b) How many bacteria will there be in 4 hours?
- (c) when will the number of bacteria be 250,000?

(B) Law of Natural Growth

- Solve the following problem using DE:
- A disease spreads through Doha at a rate proportional to the number of people in Doha. If 50 people are infected on the third day and 146 people are infected on the 6th day, determine
- (i) how many people will be infected on day 12?
- (ii) Relief from the UN will come only when at least 7.5% of Doha's population (currently say 850,000) is infected. When will relief come?

Example from IB

The population of mosquitoes in a specific area around a lake is controlled by pesticide.

The rate of decrease of the number of mosquitoes is proportional to the number of mosquitoes at any time *t*. Given that the population decreases from 500 000 to 400 000 in a five year period, find the time it takes in years for the population of mosquitoes to decrease by half.

Example from IB

- A certain population can be modelled by the differential equation dy/dt = kycoskt, where y is the population at time t hours and k is a positive constant.
- (a) Given that $y = y_0$ when t = 0, express y in terms of k, t and y_0.
- (b) Find the ratio of the minimum size of the population to the maximum size of the population.

(B) Law of Natural Growth

- Our previous question makes one KEY assumption about spread of disease (and population growth in general)
- Which is ….. ?????
- So now we need to adjust for more realistic population growth assumptions

(C) Newton's Law of Cooling

- Write a DE for the following scenario:
- The rate at which a hot object cools to the ambient temperature of its surroundings is proportional to the temperature difference between the body and its surroundings, given that $T(0) = T_o$
- Now solve this DE

(C) Newton's Law of Cooling

- The DE can be written as $\frac{dT}{dt} = k(T - A)$
- Where T(t) is the temperature at any given time and A is the ambient (room) temperature
- And the solution turns out to be

$$T(t) = A + (T_o - A)e^{kt}$$

(C) Newton's Law of Cooling

- In a steel mill, rod steel at 900ºC is cooled by forced air at a temperature of 20ºC. The temperature of the steel after one second is 400ºC. When will the steel reach a temperature of 40ºC?

(C) Newton's Law of Cooling

- A pie is removed from a 175ºC oven. The room temperature is 24ºC. How long will it take the pie to cool to 37ºC if it cooled 60º in the first minute?

(C) Newton's Law of Cooling

- A thermometer reading -7ºC is brought into a room kept at 23ºC. Half a minute later, the thermometer reads 8ºC. What is the temperature reading after three minutes?

(D) Logistic Equation

- A more realistic model for population growth in most circumstances, than the exponential model, is provided by the Logistic Differential Equation.
- In this case one's assumptions about the growth of the population include a maximum size beyond which the population cannot expand.
- This may be due to a space limitation, a ceiling on the food supply or the number of people concerned in the case of the spread of a rumor.

(D) Logistic Equation

- One then assumes that the growth rate of the population is proportional to number of individuals present, P(t), but that this rate is now constrained by how close the number P(t) is to the maximum size possible for the population, M.
- A natural way to include this assumption in your mathematical model is
- $P'(t) = k \cdot P(t) \cdot (M - P(t))$.

(D) Logistic Equation

- The differential equation $\frac{dP}{dt} = kP(M-P)$

can be interpreted as follows (where M refers to the total population (max if you will) and P is obviously the present number of people infected at time, t): the growth rate of the disease is proportional to the product of the number of people who are infected and the number of people who aren't

(D) Logistic Equation

- So to solve this equation, we separate our variables and try to integrate …..

$$\frac{dP}{dt} = kP(M-P)$$

$$\frac{1}{P(M-P)}dP = kdt$$

$$\int \frac{1}{P(M-P)}dP = \int kdt$$

(D) Logistic Equation

- Now we notice something new in our separable DE ➔ the denominator of the fraction is a product!!
- This gets into a technique called partial fractions (AARRGGHHH), so we will avoid it and simply use some tricks on the TI-89

(D) Logistic Equation

- Consider the product $\frac{2}{x(x-2)}$
- So what two fractions where added together to form this product fraction ==> i.e.

$$\frac{2}{x(x-2)}=\frac{a}{x}+\frac{b}{x-2}$$

(D) Logistic Equation

- Consider the product $\frac{2}{x(x-2)}$
- So what two fractions where added together to form this product fraction ==> i.e. $\frac{2}{x(x-2)}=\frac{a}{x}+\frac{b}{x-2}$
- You are welcome to run through the algebra yourself, but why not simply expand $\frac{2}{x(x-2)}$

$$\frac{2}{x(x-2)}=\frac{1}{x-2}+\frac{-1}{x}=\frac{1}{x-2}-\frac{1}{x}$$

(D) Logistic Equation

- So back to our DE $\int\frac{1}{P(M-P)}dP=\int kdt$
- We get $\frac{1}{P(M-P)}=\frac{1/M}{P}-\frac{1/M}{M-P}$
- So this will simplify our integration because the integral of a sum/difference is simply the sum/difference of integrals …..

(D) Logistic Equation

- So then integrating gives us

$$\int\frac{1}{P(M-P)}dP=\int kdt$$

$$\int\left(\frac{1/M}{P}-\frac{1/M}{M-P}\right)dP=kt$$

$$\int\frac{1/M}{P}dP-\int\frac{1/M}{M-P}dP=kt$$

$$\frac{1}{M}\ln P-\frac{1}{M}\ln(M-P)=kt+C$$

(D) Logistic Equation

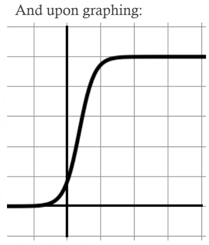
- And upon simplification, we get

$$\frac{1}{M}\ln P-\frac{1}{M}\ln(M-P)=kt+C$$

$$P(t)=\frac{M}{1+Ce^{-kMt}}=\frac{M}{1+Ce^{-Kt}}$$

- Which we call a logistic function

- And upon graphing:



(E) Examples

- Fish biologists put 200 fish into a lake whose carrying capacity is estimated to be 10,000. The number of fish quadruples in the first year

(a) Determine the logistic equation
(b) How many years will it take to reach 5000 fish?
(c) When there are 5000 fish, fisherman are allowed to catch 20% of the fish. How long will it take to reach a population of 5000 fish again?
(d) When there are 7000 fish, another 20% catch is allowed. How long will it take for the population to return to 7000?

Mixing Problems

Example 1. A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

How much salt will there be in the tank after 30 minutes?

Example 2. A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 l of water.

How much salt is in the tank after 30 minutes?

Example 3. A tank has pure water flowing into it at 12 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

Mixing Problems – Worked Example

- A tank contains 20 kg of salt dissolved in 5000 L of water.
  - Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 25 L/min.
  - The solution is kept thoroughly mixed and drains from the tank at the same rate.
  - How much salt remains in the tank after half an hour?

MIXING PROBLEMS

- Let $y(t)$ be the amount of salt (in kilograms) after t minutes.
- We are given that $y(0) = 20$ and we want to find $y(30)$.
  - We do this by finding a differential equation satisfied by $y(t)$.

MIXING PROBLEMS

- Note that dy/dt is the rate of change of the amount of salt.
- Thus, $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$

  where:
  - 'Rate in' is the rate at which salt enters the tank.
  - 'Rate out' is the rate at which it leaves the tank.

RATE IN

- We have:

$$\text{rate in} = \left(0.03\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right)$$
$$= 0.75\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS

- The tank always contains 5000 L of liquid.
  - So, the concentration at time t is $y(t)/5000$ (measured in kg/L).

RATE OUT

- As the brine flows out at a rate of 25 L/min, we have:

$$\text{rate out} = \left(\frac{y(t)}{5000}\frac{\text{kg}}{\text{L}}\right)\left(25\frac{\text{L}}{\text{min}}\right) = \frac{y(t)}{200}\frac{\text{kg}}{\text{min}}$$

MIXING PROBLEMS

- Thus, from Equation 5, we get:

$$\frac{dy}{dt} = 0.75 - \frac{y(t)}{200} = \frac{150 - y(t)}{200}$$

  - Solving this separable differential equation, we obtain:

$$\int \frac{dy}{150 - y} = \int \frac{dt}{200}$$

$$-\ln|150 - y| = \frac{t}{200} + C$$

MIXING PROBLEMS

- Since $y(0) = 20$, we have:
  $-\ln 130 = C$

So, $-\ln|150 - y| = \dfrac{t}{200} - \ln 130$

MIXING PROBLEMS

- Therefore, $|150 - y| = 130e^{-t/200}$
  - $y(t)$ is continuous and $y(0) = 20$, and the right side is never 0.
  - We deduce that $150 - y(t)$ is always positive.

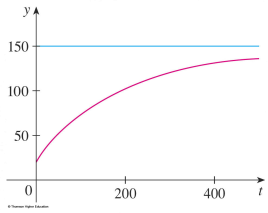
MIXING PROBLEMS

- Thus, $|150 - y| = 150 - y$.
- So, $y(t) = 150 - 130e^{-t/200}$
  - The amount of salt after 30 min is:

$$y(30) = 150 - 130e^{-30/200} \approx 38.1 \text{ kg}$$

MIXING PROBLEMS

- Here's the graph of the function $y(t)$ of Example 6.
  - Notice that, as time goes by, the amount of salt approaches 150 kg.



Further Mixing Problems

Problem 9. A cylindrical bucket has height 15 inches and cross-sectional area 200 square inches, and is initially full of water. A hole is opened in the bottom; the hole has area .5 square inches. The velocity of water exiting the bucket is $v(t) = \sqrt{Kh}$, where h is the height of the water in the bucket, and $K = 5in/s^2$.

(a) Let V be the volume of water in the bucket at time t. What is $\frac{dV}{dt}$? Your answer can have h in it.

(b) Substitute $V(t) = 200h(t)$ into this differential equation. What is $\frac{dh}{dt}$? Your answer should only be in terms of h.

(c) Find the general solution to the differential equation from part (b).

(d) What is the initial condition for this differential equation? Find the particular solution with this condition.

(e) How long does it take to empty the bucket?

Further Mixing Problems

Problem 10. A tank holds 600 gallons, and it is initially half full of pure water. Vinegar is pumped into the tank at a rate of 30 gallons per minute, and well-mixed liquid is pumped out at a rate of 15 gallons per minute.

(a) How much liquid, in gallons, is in the tank at time t?

(b) Let v be the amount of vinegar in the tank at time t, in gallons. What is the concentration of vinegar in the tank at time t, in terms of v?

(c) Write a differential equation for the amount of vinegar, in gallons, in the tank at time t.

(d) Solve the equation in part (c).

(e) How long does it take for the tank to fill up completely?

(f) When the tank is full, how much vinegar is in it?

Further Mixing Problems

Problem 11. The population of deer in a forest at time t in years is given by $P(t)$. Each year, 1 new deer is born for every 10 existing deer. However, hunters are allowed to kill H deer per year. Initially, the forest contains $P_0 = 600$ deer.

(a) Set up a differential equation governing the behavior of $P(t)$. This should be of the form $P' =$ (deer gained per year) – (deer lost per year).

(b) For what value of H is $P' = 0$? We'll call this the *critical value* of H.

(c) Solve the equation in part (a). (Your solution will still have H in it, but should not have any other constants.)

(d) Now plug in a value of H that's smaller than the critical value. Graph your solution with this value of H. What happens over a long period of time?

(e) Try the same thing with a value of H larger than the critical value. What happens over a long time now?

(f) Think about what these graphs mean.

(g) What value of P_0 makes the critical value equal to 75? (This value represents the number of deer that the forest must initially contain in order to make it sustainable to hunt 70 deer per year.)

Further Mixing Problems

Problem 12. A tank is initially filled with 1250 gallons of water, in which 100 pounds of salt are dissolved. It is filled with pure water from a pipe that pumps in 40 gallons per minute. Additionally, 10 gallons of well-mixed water per minute drain out of the tank. When the tank has 2000 gallons in it, how much salt is still present?