Lesson 55 – Differential Equations

Calculus – Santowski

Lesson Objectives

- 1. Become familiar with a definition of and terminology involved with differential equations
- 2. Solve differential equations with and without initial conditions
- 3. Apply differential equations in a variety of real world applications

Example (#1)

The acceleration of gravity is constant (near the surface of the earth). So, for falling objects:

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$$\frac{dv}{dt} = g$$

Since velocity is the rate of change of position, we could write a second order equation:

$$\frac{d^2x}{dt^2} = g$$

Example (#2)

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$$\frac{dv}{dt} = g - kv^2 \quad \text{or} \quad \frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2$$

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$$\frac{dy}{dt} = -k\,y$$

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In the presence of abundant resources (food and space), the organisms in a population will reproduce as fast as they can - this means that the rate of increase of the population will be proportional to the population itself:

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$$\frac{dP}{dt} = kP$$

..and another

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$$\boxed{\frac{dB}{dt} = kB}$$

More realistic situations for the last couple of problems

For populations: An ecosystem may have a maximum capacity to support a certain kind of organism (we're worried about this very thing for people on the planet!).

In this case, the rate of change of population is proportional *both* to the number of organisms present and to the amount of *excess capacity in the environment* (overcrowding will cause the population growth to decrease).

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$$\frac{dP}{dt} = kP(P_{\max} - P)$$

(A) Definitions

- A differential equation is any equation which contains derivatives
- Ex: $\frac{dy}{dx} = x \sin y$
- Ex: Newton's second law of motion can also be rewritten as a differential equation:
- $F = ma$ which we can rewrite as

$$F = m\frac{dv}{dt} \quad \text{or} \quad F = m\frac{d^2s}{dt^2}$$

(A) Definitions

- The order of a differential equation refers to the largest derivative present in the DE
- A solution to a differential equation on a given interval is any function that satisfies the differential equation in the given interval

Differential Equations

Definition A differential equation is an equation involving derivatives of an unknown function and possibly the function itself as well as the independent variable.

Example $y' = \sin(x), \quad (y')^4 - y^2 + 2xy - x^2 = 0, \quad y'' + y^3 + x = 0$

1st order equations — 2nd order equation

Definition The order of a differential equation is the highest order of the derivatives of the unknown function appearing in the equation

In the simplest cases, equations may be solved by direct integration.

Examples

$$y' = \sin(x) \Rightarrow y = -\cos(x) + C$$

$$y'' = 6x + e^x \Rightarrow y' = 3x^2 + e^x + C_1 \Rightarrow y = x^3 + e^x + C_1 x + C_2$$

Observe that the set of solutions to the above 1st order equation has 1 parameter, while the solutions to the above 2nd order equation depend on two parameters.

(B) Example

- Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to the second order DE $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$
- Why is the restriction $x > 0$ necessary?
- Are the equations listed below also solutions??

$$y(x) = x^{\frac{1}{2}} \text{ or } y(x) = -9x^{-\frac{3}{2}} \text{ or } y(x) = -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}$$

(C) Initial Conditions

- As we saw in the last example, a DE will have multiple solutions (in terms of many functions that satisfy the original DE)
- So to be able to identify or to solve for a specific solution, we can given a condition (or a set of conditions) that will allow us to identify the one single function that satisfies the DE

(D) Example – IVP

- Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to the second order DE $4x^2y'' + 12xy' + 3y = 0$ given the initial conditions of

$$y(4) = \frac{1}{8} \text{ and } y'(4) = -\frac{3}{64}$$

- What about the equation

$$y(x) = -9x^{-\frac{3}{2}} + 7x^{-\frac{1}{2}}$$

(D) Examples

- Ex 1. Show that $y(x) = \dfrac{3\ln x}{2} - 2x + 5$ is a general solution to the DE $2xy' + 4x = 3$
- Ex 2. What is the solution to the IVP $2xy' + 4x = 3$ and $y(1) = -4$

(E) Further Examples

- Ex 1. What is the solution to the first order differential equation (FODE)

$$\frac{dy}{dx} = 2x$$

- Sketch your solution on a graph.
- Then solve the IVP given the initial condition of (-1,1)

(E) Further Examples

- Ex 2. Solve the IVP

$$\frac{dy}{dx} = \sin x - \frac{1}{x} \text{ and } (e, 1 - \cos(e))$$

- How would you verify your solution?

(E) Further Examples

- Ex 3. What is the solution to the first order differential equation (FODE)

$$\frac{dy}{dx} = \sec^2 x + 2x + 5$$

- Then solve the IVP given the initial condition of

$$(\pi, \pi^2 + 5\pi + 2)$$

(E) Further Examples

- Ex 4. What is the solution to the first order differential equation (FODE)

$$\frac{dy}{dx} = e^x - 6x^2$$

- Then solve the IVP given the initial condition of

$$(1,0)$$

(E) Further Examples

- Ex 5. What is the solution to the first order differential equation (FODE)

$$f'(x) = e^{-x^2} \text{ and } f(7) = 3$$

- (Note that this time our solution simply has an integral in the solution – FTC)

(F) Motion Examples

- If a body moves from rest so that its velocity is proportional to the time elapsed, show that $v^2 = 2as$

where s = displacement, t is time and v is velocity and a is a constant (representing ….?)

(F) Motion Examples

- A particle is accelerated in a line so that its velocity in m/s is equal to the square root of the time elapsed, measured in seconds. How far does the particle travel in the first hour?

(F) Motion Examples

- A stone is tossed upward with a velocity of 8 m/s from the edge of a cliff 63 m high. How long will it take for the stone to hit the ground at the foot of the cliff?