Lesson 50 – Area Between Curves

HL Math- Santowski

FAST FIVE

- True or false & explain your answer:
  $$\int_0^{2\pi} \sec^2 x \, dx = \tan x \Big|_0^{2\pi} = 0 - 0 = 0$$

Fast Five

- 1. Graph $v_1(t) = \begin{cases} t & 0 < t \le 3 \\ 3 & 3 < t \le 7 \end{cases}$
- 2. Integrate $\int -\frac{1}{4}(t-4)^2 + 4 \ dt$
- 3. Integrate $\int \left(-2\cos(t) + 5\right) dt$ $\int \left(-1.5\sin(0.8t) + 4\right) dt$
- 4. Evaluate $\int_1^{10} \left(-2\cos(t) + 5\right) dt$ $\int_3^6 \left(-\frac{1}{4}(t-4)^2 + 4\right) dt$

Lesson Objectives

- 1. Determine total areas under curves
- 2. Apply definite integrals to a real world problems

(A) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. An object starts at the origin and moves along the x-axis with a velocity $v(t) = 10t - t^2$ for $0 \le t \le 10$
- (a) What is the position of the object at any time, t ?
- (b) What is the position of the object at the moment when it reaches its maximum velocity?
- (c) How far has the object traveled before it starts to decelerate?

(A) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- 1. For the velocity functions given below for a particle moving along a line, determine the distance traveled and displacement of the particle:
- (a) $v(t) = 3t - 5, 0 \le t \le 3$
- (b) $v(t) = t^2 - 2t - 8, 1 \le t \le 6$
- 2. The acceleration function and the initial velocity are given for a particle moving along a line. Determine (a) the velocity at time t and (b) the distance traveled during the given time interval:
- (a) $a(t) = t + 4,\ v(0) = 5,\ 0 \le t \le 10$
- (b) $a(t) = 2t + 3, v(0) = -4,\ 0 \le t \le 3$

(A) APPLICATIONS OF DEFINITE INTEGRALS – MOTION PROBLEMS

- Two cars, who are beside each other, leave an intersection at the same instant. They travel along the same road. Car A travels with a velocity of $v(t) = t^2 - t - 6$ m/s while Car B travels with a velocity of $v(t) = 0.5t + 2$ m/s.
- (a) What are the initial velocities of the cars?
- (b) How far has each car gone after 4 seconds have elapsed?
- (c) When are the two cars beside each other again (i.e. when does the trailing car catch up to the leading car?)

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(B) General formula

- To find the area between 2 curves, we use the general formula

$$\int_a^b \left(f(x) - g(x)\right) dx$$

- Given that $y = f(x)$ and $y = g(x)$ are continuous on [a,b] and that $f(x) \geq g(x)$ on $[a,b]$

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(C) Examples

- Sketch the curve of $f(x) = x^3 - x^4$ between x = 0 and x = 1.
- (a) Draw a vertical line at $x = k$ such that the region between the curve and axis is divided into 2 regions of equal area. Determine the value of k.
- (b) Draw a horizontal line at $y = h$ such that the region between the curve and axis is divided into 2 regions of equal area. Estimate the value of h. Justify your estimation.

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(C) Examples

- (a) Find the area bounded by
  $f(x) = -x^2 + 1$, $g(x) = 2x + 4$, x = -1, and $x = 2$
- (b) Find the area between the curves
  $h(x) = x^2 - 2x$ and $k(x) = x$ on $[0,4]$
- (c) Find the region enclosed by $y = \sqrt{x}$ and $y = x^3$
- (d) Find the region enclosed by $y = \sin x$ and $y = \pi x - x^2$

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(D) Area Under VT Graphs

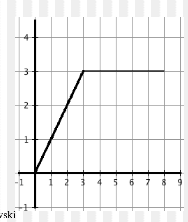
- Let's deal with a journey of two cars as illustrated by their VT graphs
- We will let the two cars start at the same spot
- Here is the graph for Car 1
- Highlight, calculate & interpret:

(i) $v_1(4)$ (ii) $\int_0^3 v_1(t)dt$

(iii) $\int_0^5 v_1(t)dt$ (iv) $\int_3^6 v_1(t)dt$

- The equation is

$$v_1(t) = \begin{cases} t & 0 < t \leq 3 \\ 3 & 3 < t \leq 7 \end{cases}$$



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(D) Area Under VT Graphs

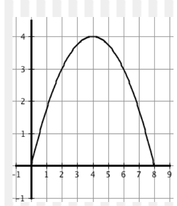
- Let's deal with a journey of two cars as illustrated by their VT graphs
- We will let the two cars start at the same spot
- Here is the graph for Car II
- Calculate, highlight & interpret:

(i) $v_2(4)$ (ii) $\int_0^3 v_2(t)dt$

(iii) $\int_0^5 v_2(t)dt$ (iv) $\int_3^6 v_2(t)dt$

- The equation is

$$v_2(t) = -\frac{1}{4}(t-4)^2 + 4$$



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(D) Area Under VT Graphs

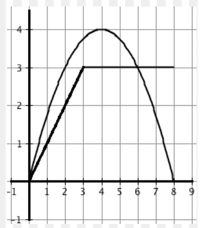
- Now let's add a new twist to this 2 car problem:
- Highlight, calculate and interpret:

(i) $\int_{0}^{3}\left(v_2(t)-v_1(t)\right)dt$

(ii) $\int_{3}^{6}\left(v_2(t)-v_1(t)\right)dt$

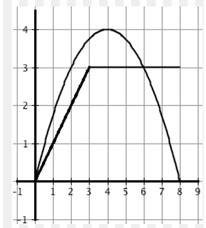
(iii) $\int_{6}^{8}\left(v_1(t)-v_2(t)\right)dt$

- Here are the two graphs:



(D) Area Under VT Graphs

- Final question about the 2 cars:
- When do they meet again (since we have set the condition that they started at the same point)
- Explain how and why you set up your solution
- Here are the graphs again:

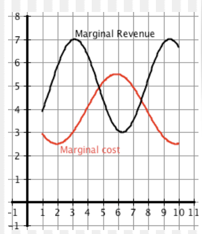


(E) Economics Applications

- Now let's switch applications to economics
- Our two curves will represent a marginal cost function and a marginal revenue function
- The equations are:

$$MR(t) = -2\cos(t) + 5\ ,\quad 1 \le t \le 10$$
$$MC(t) = -1.5\sin(0.8t) + 4\ ,\quad 1 \le t \le 10$$

- Here are the two curves



(E) Economics Applications

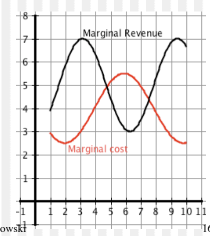
- Highlight, calculate and evaluate the following integrals:

(i) $\int_{1}^{4} \mathrm{MR}(t)dt$

(ii) $\int_{1}^{4} \mathrm{MC}(t)dt$

(iii) $\int_{1}^{4} (\mathrm{MR}(t) - MC(t))dt$

- Here are the 2 functions:



(E) Economics Applications

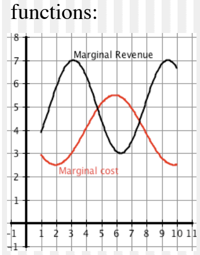
- Highlight, calculate and evaluate the following integrals:

(i) $\int_{4.75}^{7.54} \mathrm{MR}(t)dt$

(ii) $\int_{4.75}^{7.54} \mathrm{MC}(t)dt$

(iii) $\int_{4.75}^{7.54} (\mathrm{MC}(t) - \mathrm{MR}(t))dt$

- Here are the 2 functions:

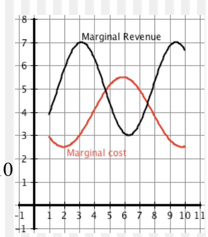


(E) Economics Applications

- So the final question then is how much profit did the company make between months 1 and 10?

$$MR(t) = -2\cos(t) + 5\ ,\quad 1 \le t \le 10$$
$$MC(t) = -1.5\sin(0.8t) + 4\ ,\quad 1 \le t \le 10$$

- The 2 functions



Lesson 50b - AVERAGE VALUE OF A FUNCTION

Calculus - Santowski

Lesson Objectives

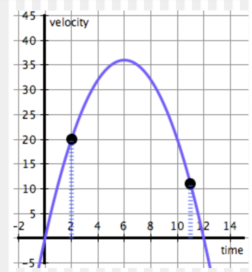
- 1. Understand average value of a function from a graphic and algebraic viewpoint
- 2. Determine the average value of a function
- 3. Apply average values to a real world problems

Fast Five

- 1. Find the average of 3,7,4,12,5,9
- 2. If you go 40 km in 0.8 hours, what is your average speed?
- 3. How far do you go in 3 minutes if your average speed was 600 mi/hr
- 4. How long does it take to go 10 kilometers at an average speed of 30 km/hr?

(A) Average Speed

- Suppose that the speed of an object is given by the equation $v(t) = 12t-t^2$ where v is in meters/sec and t is in seconds. How would we determine the average speed of the object between two times, say t = 2 s and t = 11 s



(A) Average Velocity

- So, let's define average as how far you've gone divided by how long it took or more simply displacement/time
- which then means we need to find the displacement. HOW??
- We can find total displacement as the area under the curve and above the x-axis => so we are looking at an integral of

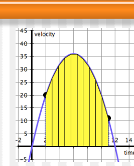
$$s = \int_2^{11} (12t - t^2)dt$$

- Upon evaluating this definite integral, we get

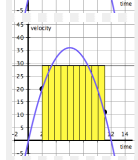
$$s = \int_2^{11} (12t - t^2)dt = 6t^2 - \frac{t^3}{3}\Big|_2^{11} = 261 \text{ m}$$

(A) Average Value

- If the total displacement is 261 meters, then the average speed is 261 m/(11-2)s = 29 m/s



- We can visualize our area of 261 m in a couple of ways: (i) area under v(t) or (ii) area under v = 29 m/s



(A) Average Velocity

- So our "area" or total displacement is seen from 2 graphs as (i) area under the original graph between the two bounds and then secondly as (ii) the area under the horizontal line of v = 29 m/s or rather under the horizontal line of the average value
- So in determining an average value, we are simply trying to find an area under a horizontal line that is equivalent to the area under the curve between two specified t values
- So the real question then comes down to "how do we find that horizontal line?"

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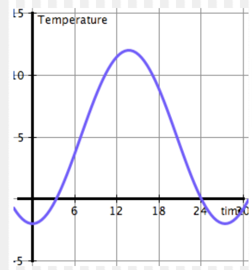
(B) Average Temperature

- So here is a graph showing the temperatures of Toronto on a minute by minute basis on April 3rd.
- So how do we determine the average daily temperature for April 3rd?



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(B) Average Temperature

- So to determine the average daily temperature, we could add all 1440 (24 x 60) times and divide by 1440 ➔ possible but tedious
- What happens if we extended the data for one full year (525960 minutes/data points)
- So we need an approximation method

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(B) Average Temperature

- So to approximate:
- (1) divide the interval (0,24) into n equal subintervals, each of width $\Delta x = (b-a)/n$
- (2) then in each subinterval, choose x values, $x_1, x_2, x_3, \ldots, x_n$
- (3) then average the function values of these points: $[f(x_1) + f(x_2) + \ldots + f(x_n)]/n$
- (4) but n = (b-a)/ Δx
- (5) so $f(x_1) + f(x_2) + \ldots + f(x_n)/((b-a)/\Delta x)$
- (6) which is $1/(b-a)\ [f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x]$
- (7) so we get $1/(b-a)\Sigma f(x_i)\Delta x$

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(B) Average Temperature

- Since we have a sum of $\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$
- Now we make our summation more accurate by increasing the number of subintervals between a & b:

$$\lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

- Which is of course our integral

$$\left(\frac{1}{b-a}\right)\times\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \left(\frac{1}{b-a}\right)\times\int_a^b f(x)\,dx$$

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(B) Average Temperature

- So finally, average value is given by an integral of

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- So in the context of our temperature model, the equation modeling the daily temperature for April 3 in Toronto was

$$T(t) = 7\sin\left(\frac{5.5t}{24} + 11\right) + 5$$

- Then the average daily temp was

$$T_{ave} = \frac{1}{24}\int_0^{24}\left(7\sin\left(\frac{5.5t}{24} + 11\right) + 5\right)dt = 6.7^o$$

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(C) Examples

- Find the average value of the following functions on the given interval

$$(i)\ f(x)=x^2-2x,\ [0,3]$$
$$(ii)\ f(x)=\sin x,\ [0,\pi]$$
$$(iii)\ f(x)=e^{2x},\ [0,2]$$
$$(iv)\ f(x)=\frac{1}{x},\ [1,4]$$

(D) Mean Value Theorem of Integrals

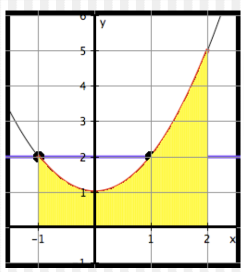
- Given the function $f(x) = 1+x^2$ on the interval [-1,2]
- (a) Find the average value of f(x)

$$f_{ave}=\frac{1}{2-(-1)}\int_{-1}^{2}\left(1+x^2\right)dx=2$$

- Question ⇒is there a number in the interval at x = c at which the function value at c equals the average value of the function?
- So we set the equation then as $f(c) = 2$ and solve $2 = 1+c^2$
- Thus, at $c = \pm 1$, the function value is the same as the average value of the function

(D) Mean Value Theorem of Integrals

- Our solution to the previous question is shown in the following diagram:



(E) Examples

- For the following functions, determine:
- (a) the average value on the interval
- (b) Determine c such that $f_c = f_{ave}$
- (c) Sketch a graph illustrating the two equal areas (area under curve and under rectangle)
-

$$(i)\ f(x)=4-x^2,\ [0,2]$$
$$(ii)\ f(x)=x\sin\left(x^2\right),\ \left[0,\sqrt{\pi}\right]$$
$$(iii)\ f(x)=x^3-x+1,\ [0,2]$$