Lesson 48 – Definite Integrals & Area under Curves

IB MATHEMATICS HL – CALCULUS

(A) Review – Area from First Principles

- We have determined the area under a curve in two ways now:
- (a) Estimated the area by determining the sum of a finite number of constructed rectangles (or trapezoids if we wanted)

$A = \sum_{i=1}^{n} f(x_i)\Delta x$ for some finite number of n

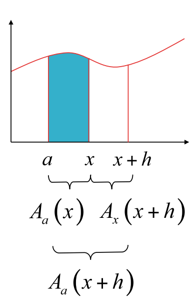
(A) Review – Area from First Principles

- We have determined the area under a curve in two ways now:
- (b) Getting exact areas using the idea of the limits and an infinite number of rectangles – which is tedious algebraically (and difficult once we get away from polynomial based functions!!)

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

(B) Area – NEW Approach?

- Just as we saw with differentiation, we move BEYOND the "first principles" and develop an easier, more convenient way to calculate our areas …….
- Here is a series of slides, explaining to you HOW we develop the NEW approach



Let $A_a(x) =$ area under the curve from a to x.

("a" is a constant)

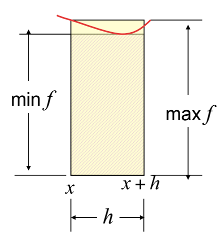
Then:

$$A_a(x) + A_x(x+h) = A_a(x+h)$$

$$A_x(x+h) = A_a(x+h) - A_a(x)$$

→

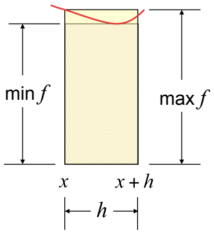
Let's consider the area of $A_x(x+h)$



The area of a rectangle drawn under the curve would be less than the actual area under the curve.

The area of a rectangle drawn above the curve would be more than the actual area under the curve.

→



The area of a rectangle drawn under the curve would be less than the actual area under the curve.

The area of a rectangle drawn above the curve would be more than the actual area under the curve.

short rectangle ≤ area under curve ≤ tall rectangle

$$h \cdot \min f \le A_a(x+h) - A_a(x) \le h \cdot \max f$$

$$\min f \le \frac{A_a(x+h) - A_a(x)}{h} \le \max f$$

$$\min f \le \frac{A_a(x+h) - A_a(x)}{h} \le \max f$$

As h gets smaller, min f and max f get closer together.

$$\lim_{h\to 0} \frac{A_a(x+h) - A_a(x)}{h} = f(x)$$

$$\frac{d}{dx} A_a(x) = f(x)$$

$$A_a(x) = F(x) + c$$

$$A_a(a) = F(a) + c$$

initial value

$$0 = F(a) + c$$

$$-F(a) = c$$

Take the anti-derivative of both sides to find an explicit formula for area.

$$\min f \le \frac{A_a(x+h) - A_a(x)}{h} \le \max f$$

As h gets smaller, min f and max f get closer together.

$$\lim_{h\to 0} \frac{A_a(x+h) - A_a(x)}{h} = f(x)$$

$$\frac{d}{dx} A_a(x) = f(x)$$

$$A_a(x) = F(x) + c$$

$$A_a(a) = F(a) + c$$

$$0 = F(a) + c$$

$$-F(a) = c$$

$$\boxed{A_a(x) = F(x) - F(a)}$$

Area under curve from a to x ➔ antiderivative at x minus antiderivative at a.

$$\text{Area} = A(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\text{Area} = A(x) = F(b) - F(a)$$

$$\text{Area} = A(x) = \int_a^b f(x)dx$$

Examples #1

- Find the area between the x-axis and the curve $f(x)=x^2+1$ on the interval $-1 < x < 2$

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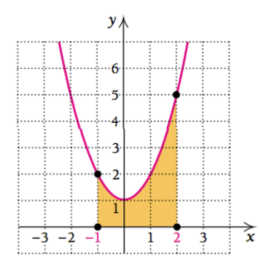

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Examples #1

- Find the area between the x-axis and the curve f(x)=x² +1 on the interval -1 < x < 2

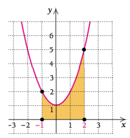
Solution In this case, $f(x) = x^2 + 1$, $a = -1$, and $b = 2$.

1) Find any antiderivative $F(x)$ of $f(x)$. We choose the simplest one:

$$F(x) = \frac{x^3}{3} + x.$$

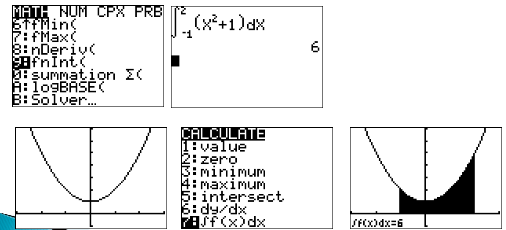
2) Substitute 2 and −1 and find the difference $F(2) - F(-1)$:

$$\begin{aligned} F(2) - F(-1) &= \left[\frac{2^3}{3} + 2\right] - \left[\frac{(-1)^3}{3} + (-1)\right] \\ &= \frac{8}{3} + 2 - \left[\frac{-1}{3} - 1\right] \\ &= \frac{8}{3} + 2 + \frac{1}{3} + 1 \\ &= 6. \end{aligned}$$



Examples #1 - Using TI-84

- Find the area between the x-axis and the curve f(x)=x² +1 on the interval -1 < x < 2



Examples #2

- Find the area under the curves specified below:

(a) $f(x) = \dfrac{1}{x+3}$ on $[-2,1]$

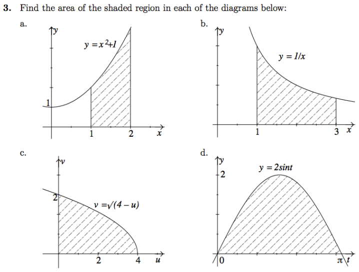
(b) $g(x) = 1 + \sin 2x$ on $\left[0, \dfrac{\pi}{4}\right]$

(c) $h(x) = 4 - x + x^3$ on $[-1,2]$

(d) $m(x) = e^x - x$ on $\left[-\dfrac{1}{e}, e\right]$

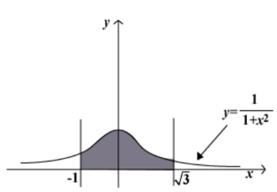
Examples #3

3. Find the area of the shaded region in each of the diagrams below:



Examples #4

Find the area bounded by $y = \frac{1}{1+x^2}$, $x = -1$, $x = \sqrt{3}$ and the x-axis i.e. find the area of shaded region R in the picture below:



Examples #4

Find the area bounded by $y = \frac{1}{1+x^2}$, $x = -1$, $x = \sqrt{3}$ and the x-axis i.e. find the area of shaded region R in the picture below:

$$\begin{aligned} \text{Area} &= \int_{-1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \left[\text{Arctan}x\right]_{-1}^{\sqrt{3}} \\ &= \text{Arctan}\sqrt{3} - \text{Arctan}(-1) \\ &= \frac{\pi}{3} - \left(-\frac{\pi}{4}\right) \\ &= \frac{7\pi}{12}. \end{aligned}$$

Example #5

- For the following definite integrals:
- (i) Evaluate without the TI-84
- (ii) Evaluate using a graph on the TI-84
- (iii) Interpret your results

(a) $\int_0^2 -x^2 dx$

(b) $\int_{-1}^2 (x^2 - 1) dx$

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Example #6

Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.

Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.

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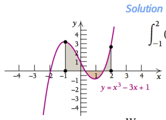
Solution

- (a)

Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.

EXAMPLE 11 Evaluate: $\int_{-1}^{2} (x^3 - 3x + 1)\, dx$. Interpret the results in terms of area.

Solution We have



$$\int_{-1}^{2} (x^3 - 3x + 1)\, dx = \left[\frac{x^4}{4} - \frac{3}{2}x^2 + x\right]_{-1}^{2}$$
$$= \left(\frac{2^4}{4} - \frac{3}{2}\cdot 2^2 + 2\right) - \left(\frac{(-1)^4}{4} - \frac{3}{2}\cdot(-1)^2 + (-1)\right)$$
$$= \left(\frac{16}{4} - 6 + 2\right) - \left(\frac{1}{4} - \frac{3}{2} - 1\right) = 0 - \left(-2\tfrac{1}{4}\right) = 2\tfrac{1}{4}.$$

We can graph the function $f(x) = x^3 - 3x + 1$ over the interval and shade the area between the curve and the x-axis. The sum of the areas above the axis minus the area below is $2\frac{1}{4}$.

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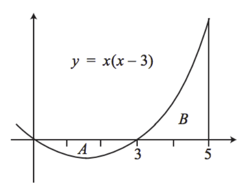
HINT:

- (b)

Example

Find the area between the curve $y = x(x-3)$ and the ordinates $x = 0$ and $x = 5$.



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Example #7

2. Find the area contained by the curve $y = x(x-1)(x+1)$ and the x-axis.
3. Calculate the value of
$$\int_{-1}^{1} x(x-1)(x+1)\mathrm{d}x\,.$$
Compare your answer with that obtained in question 3, and explain what has happened.
4. Calculate the value of
$$\int_{0}^{6} (4x - x^2)\mathrm{d}x\,.$$
Explain your answer.

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Example #8

1. Find the area enclosed by the graph of $y = 3x^2(x-4)$ and the x-axis.
2. i Find the value of $\int_0^{2\pi} \sin x\,dx$.
   ii Find the area enclosed between the graph of $y = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$.
3. Find the total area enclosed between the graph of $y = 12x(x+1)(2-x)$ and the x-axis.

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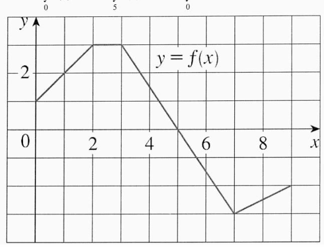
Example #9

The graph of f is shown below. Evaluate each integral by interpreting it in terms of areas.

a) $\int_0^2 f(x)\,dx$ b) $\int_0^5 f(x)\,dx$ c) $\int_5^7 f(x)\,dx$ d) $\int_0^9 f(x)\,dx$



Example #10

13. If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 2f(x)\,dx$

14. If $\int_0^9 f(x)\,dx = 37$ and $\int_0^9 g(x)dx = 16$, find $\int_0^9 [2f(x)+3g(x)]$

15. (Calculator Permitted) Use your calculator's fnInt(function to evaluate the following integrals. 3 decimials.

a) $\int_0^5 \frac{x}{x^2+4}dx$ b) $3+2\int_0^{\pi/3} \tan x dx$

Example #12 – Application

From past records a management services determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by $M'(x) = 90x^2 + 5{,}000$ where M is the total accumulated cost of maintenance for x years.

Application

From past records a management services determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by $M'(x) = 90x^2 + 5{,}000$ where M is the total accumulated cost of maintenance for x years.

Write a definite integral that will give the total maintenance cost through the seventh year. Evaluate the integral.

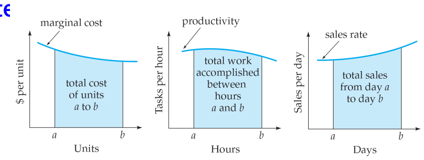
$$\int_0^7 90x^2 + 5{,}000\,dx = 30x^3 + 5{,}000x \Big|_{x=0}^{7}$$

$$= 10{,}290 + 35{,}000 - 0 - 0$$

$$= \$45{,}290$$

Total Cost of a Succession of Units

The following diagrams illustrate this idea. In each case, the *curve* represents a *rate,* and the *area under the curve,* given by the definite integral, gives the *total accumulation* at that rate



FINDING TOTAL PRODUCTIVITY FROM A RATE

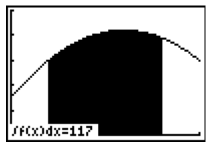
A technician can test computer chips at the rate of $-3x^2 + 18x + 15$ chips per hour (for $0 \le x \le 6$), where x is the number of hours after 9:00 a.m. How many chips can be tested between 10:00 a.m. and 1:00 p.m.?

Solution – N (t) = $-3t^2 + 18t + 15$

The total work accomplished is the integral of this rate from $t = 1$ (10 a.m.) to $t = 4$ (1 p.m.):

Use your calculator

$$\int_1^4 -3x^2 + 18x + 15\,dx$$



$$= \left(-x^3 + 9x^2 + 15x\right)\Big|_{x=1}^{4}$$

$= (-64 + 144 + 60) - (-1 + 9 + 15) = 117$

That is, between 10 a.m. and 1 p.m., 117 chips can be tested.

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