Lesson 47 - Area under Curves as Limits and Sums

IB MATHEMATICS HL - CALCULUS

Use of Calculator - Summations

- Note again summation capability of calculator
- Syntax is:

∑ (expression, variable, low, high)

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$$

$$\sum_{x=1}^{5} (x^2) \quad 55$$

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Σ(x^2,x,1,5)
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Practice Summation

- Try these

$$\sum_{k=1}^{50} 3 =$$

$$\sum_{k=1}^{5} (k+1) =$$

$$\sum_{k=1}^{40} (k-1)^2$$

The Area Problem Background SKILL: Sums of Powers

- Find the sum of the following power series:

(i) $c + c + c + c + \ldots + c = \sum_{i=1}^{n} c = ??$

(ii) $1 + 2 + 3 + 4 + \ldots + k = \sum_{i=1}^{n} i = ??$

(iii) $1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \sum_{i=1}^{n} i^2 = ??$

(iv) $1^3 + 2^3 + 3^3 + 4^3 + \ldots k^3 = \sum_{i=1}^{n} i^3 = ??$

The Area Problem Background SKILL: Sums of Powers

- We have the following summation formulas to use in our simplification:

$$\sum_{i=1}^{n} c = c + c + c + c + \ldots + c = n \times c$$

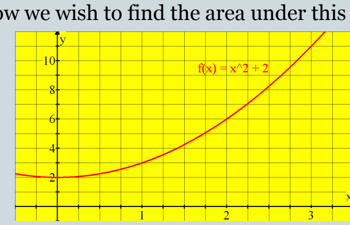
$$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 + \ldots + k = \frac{1}{2}\left(n^2 + n\right) = \frac{n(n-1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{1}{6}\left(2n^3 + 3n^2 + n\right) = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots k^3 = \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = \frac{n^2(n+1)^2}{4}$$

(A) The Area Problem - REVIEW

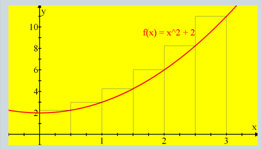
- Let's work with f(x) = x² + 2 and use a specific interval of [0,3]
- Now we wish to find the area under this curve



(A) The Area Problem – An Example

- To estimate the area under the curve, we will divide the are into simple rectangles as we can easily find the area of rectangles ➔ A = l × w
- Each rectangle will have a width of Δx which we calculate as $(b - a)/n$ where b represents the higher bound on the area (i.e. x = 3) and a represents the lower bound on the area (i.e. x = 0) and n represents the number of rectangles we want to construct
- The height of each rectangle is then simply calculated using the function equation
- Then the total area (as an estimate) is determined as we sum the areas of the numerous rectangles we have created under the curve
- $A_T = A_1 + A_2 + A_3 + \ldots + A_n$
- We can visualize the process on the next slide

(A) The Area Problem – An Example

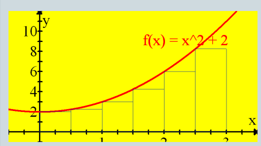


- We have chosen to draw 6 rectangles on the interval [0,3]
- $A_1 = \frac{1}{2} \times f(\frac{1}{2}) = 1.125$
- $A_2 = \frac{1}{2} \times f(1) = 1.5$
- $A_3 = \frac{1}{2} \times f(1\frac{1}{2}) = 2.125$
- $A_4 = \frac{1}{2} \times f(2) = 3$
- $A_5 = \frac{1}{2} \times f(2\frac{1}{2}) = 4.125$
- $A_6 = \frac{1}{2} \times f(3) = 5.5$
- $A_T = 17.375$ square units
- So our estimate is 17.375 which is obviously an overestimate

(A) The Area Problem – An Example

- In our previous slide, we used 6 rectangles which were constructed using a "right end point" (realize that both the use of 6 rectangles and the right end point are arbitrary!) ➔ in an increasing function like $f(x) = x^2 + 2$ this creates an over-estimate of the area under the curve
- So let's change from the right end point to the left end point and see what happens

(A) The Area Problem – An Example



- We have chosen to draw 6 rectangles on the interval [0,3]
- $A_1 = \frac{1}{2} \times f(0) = 1$
- $A_2 = \frac{1}{2} \times f(\frac{1}{2}) = 1.125$
- $A_3 = \frac{1}{2} \times f(1) = 1.5$
- $A_4 = \frac{1}{2} \times f(1\frac{1}{2}) = 2.125$
- $A_5 = \frac{1}{2} \times f(2) = 3$
- $A_6 = \frac{1}{2} \times f(2\frac{1}{2}) = 4.125$
- $A_T = 12.875$ square units
- So our estimate is 12.875 which is obviously an under-estimate

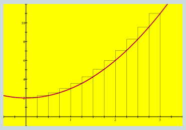
(A) The Area Problem – An Example

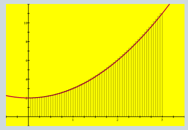
- So our "left end point" method (now called a left hand Riemann sum) gives us an underestimate (in this example)
- Our "right end point" method (now called a right handed Riemann sum) gives us an overestimate (in this example)
- Recall that ce can adjust our strategy in a variety of ways ➔ one is by adjusting the "end point" ➔ why not simply use a "midpoint" in each interval and get a mix of over- and under-estimates.
- OR we can construct trapezoids …… OR ……

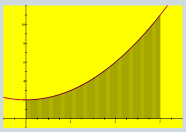
(B) The Area Problem – Expanding our Example

- Now back to our left and right Riemann sums and our original example ➔ how can we increase the accuracy of our estimate?
- We simply increase the number of rectangles that we construct under the curve
- Initially we chose 6, now let's choose a few more … say 12, 60, and 300 ….

(B) The Area Problem – Expanding our Example

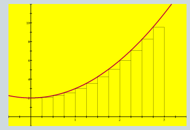


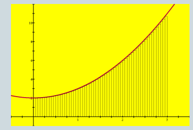


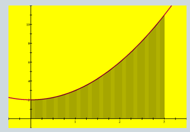


# of rectangles	Area estimate
12	16.15625
60	15.22625
300	15.04505

(B) The Area Problem – Expanding our Example







# of rectangles	Area estimate
12	13.90625
60	14.77625
300	14.95505

(C) The Area Problem - Conclusion

- We have seen the following general formula used in the preceding examples:
- $A = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_i)\Delta x + \ldots + f(x_n)\Delta x$ as we have created n rectangles
- Since this represents a sum, we can use summation

notation to re-express this formula ➔ $A = \sum_{i=1}^{n} f(x_i)\Delta x$

- So this is the formula for our Riemann sum

(D) The Area Problem – Exact Areas

- Now to make our estimate more accurate, we simply made more rectangles ➔ how many more though? ➔ why not an infinite amount (use the limit concept as we did with our tangent/secant issue in differential calculus!)
- Then we arrive at the following "formula":

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

(E) Exact Areas – Example #1

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- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using:
- (a) using 4 rectangles & RRAM
- (b) using geometry
- (c) Using an infinite number of rectangles => hence a LIMIT idea

(E) Exact Areas – Example #1

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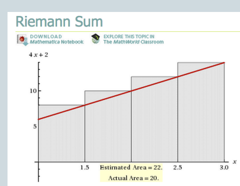
- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using 4 rectangles & RRAM:



- First, establish:

(i) $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$

(ii) $x_1 = a + \Delta x_1 = 1 + \frac{1}{2} = 1.5$

(iii) $x_2 = a + \Delta x_1 + \Delta x_2 = 1 + \frac{1}{2} + \frac{1}{2} = 2$

(iv) $x_1 = a + \Delta x_1 + \Delta x_2 + \Delta x_3 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2.5$

(v) $x_1 = a + \Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3$

in general $x_i = a + i \times \Delta x$

(E) Exact Areas – Example #1

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- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using 4 rectangles & RRAM:

$$A = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + f(x_4)\Delta x_4$$
$$A = f(1.5)\times\frac{1}{2} + f(2)\times\frac{1}{2} + f(2.5)\times\frac{1}{2} + f(3)\times\frac{1}{2}$$
$$A = 4\times\frac{1}{2} + 6\times\frac{1}{2} + 8\times\frac{1}{2} + 10\times\frac{1}{2}$$
$$A = \frac{1}{2}\times\left(4+6+8+10\right)$$
$$A = \frac{1}{2}\times 28 = 14$$

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(E) Exact Areas – Example #1

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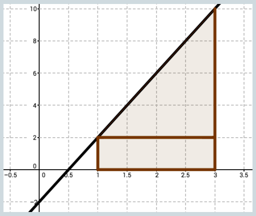
- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using geometry

$$A = A_T + A_R = \frac{1}{2}bh + lw$$
$$A = \frac{1}{2}\times 2\times\left(f(3)-f(1)\right) + \left(3-1\right)\left(f(1)\right)$$
$$A = \frac{1}{2}\times 2\times\left(10-2\right) + 2\left(2\right)$$
$$A = 8+4 = 12$$



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(E) Exact Areas – Example #1

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- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using limits of # of rectangles:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$\text{where } \Delta x = \frac{b-a}{n} = \frac{2}{n};$$
$$\text{where } x_i = a + i\Delta x = 1 + \frac{2i}{n};$$
$$\text{where } f(x_i) = f\left(1+\frac{2i}{n}\right) = 4\left(1+\frac{2i}{n}\right) - 2 = 2 + \frac{8i}{n}$$

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(E) Exact Areas – Example #1

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- Find the area under the function y = 4x – 2 between x = 1 and x = 3 using limits of # of rectangles:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)\left(\frac{2}{n}\right)$$
$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}+\frac{16i}{n^2}\right)$$
$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{n}\right) + \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{16i}{n^2}\right) = \lim_{n\to\infty}\left(\frac{4}{n}\sum_{i=1}^{n}1\right) + \lim_{n\to\infty}\left(\frac{16}{n^2}\right)\sum_{i=1}^{n} i$$
$$A = \lim_{n\to\infty}\left(\frac{4}{n}\times n\right) + \lim_{n\to\infty}\left(\frac{16}{n^2}\times\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left(4\right) + \lim_{n\to\infty}\left(\frac{16n^2+16n}{2n^2}\right)$$
$$A = \left(4\right) + \lim_{n\to\infty}\left(8+\frac{8}{n}\right) = 4+8 = 12$$

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(E) Exact Areas – Example #2

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- Find the EXACT area under the function y = 2x + 3 between x = 1 and x = 3
- For complete solution & explanation, watch video:
- https://www.youtube.com/watch?v=bw23lWXpAlc

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(E) Exact Areas – Example #3

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- Find the exact area under the curve of y = x² between x = 0 and x = 1
- VIDEO LINK to worked soln:
- https://www.youtube.com/watch?v=ebCVej2wlCo

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(E) Exact Areas – Example #4

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- Find the EXACT area under the function y = 1 – x^2 between x = 0 and x = 1
- For solution, see pages 4-5 from https://www3.nd.edu/~apilking/Math10550/Lectures/24.%20Areas%20and%20Distances.pdf
- Here are other links to worked & explained examples for:
  - (i) y = 9 – x^2 as a video and as notes
  - (ii) y = 4 – x^2 as a video

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(E) Exact Areas – Example #5

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- Find the EXACT area under the function y = x^3 between x = 1 and x = 0
- Here is a link to the detailed steps of the solution:
- http://goblues.org/faculty/kollathl/files/2010/08/Finding-Area-Using-Infinite-Rectangles.pdf
- And here is a link to a video showing the process for f(x) = 64 – x^3 ➔ https://www.youtube.com/watch?v=DVmF0eyARSc

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(E) Exact Areas – Example #6

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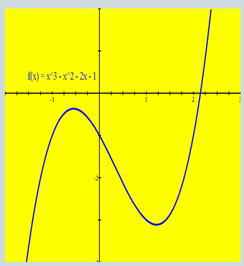
- The formula we will use is
  $$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$
- The function will be f(x) = x^3 – x^2 - 2x - 1 on [0,2]
- Which we can graph and see ➔



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(E) Exact Areas – Example #6

- So our formula is: $A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$
- Now we need to work out just what f(x_i) and Δx are equal to so we can sub them into our formula:
- Δx = (*b*-*a*)/*n* = (2-0)/n = 2/n
- x_i simply refers to the any endpoint on any one of the many rectangles ➔ so let's work with the *i*th endpoint on the *i*th rectangle ➔ so in general, x_i = *a* + iΔx

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(E) Exact Areas – Example #6

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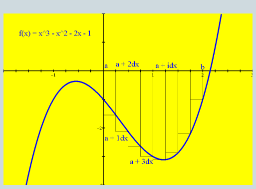

- I am using *a* + i*dx* on the diagram to represent the ith rectangle, which is iΔx units away from *a*
- Then the height of this rectangle is f(a + iΔx)

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(E) Exact Areas – Example #6

- Now back to the formula in which Δx = 2/n and x_i = 0 + i Δx or x_i = 2i/n

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(x_i^3 - x_i^2 - 2x_i - 1\right)\left(\frac{2}{n}\right)$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)^2 - 2\left(\frac{2i}{n}\right) - 1\right)\left(\frac{2}{n}\right)$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\left(\frac{16i^3}{n^4}\right) - \left(\frac{8i^2}{n^3}\right) - \left(\frac{8i^1}{n^2}\right) - \left(\frac{2}{n}\right)\right)$$

$$A = \lim_{n \to \infty} \left[\frac{16}{n^4}\sum_{i=1}^{n} i^3 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i^1 - \frac{2}{n}\sum_{i=1}^{4} 1\right]$$

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(E) Exact Areas – Example #6

- Now we simply substitute our appropriate power sums:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty}\left[\frac{16}{n^4}\sum_{i=1}^{n} i^3 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{8}{n^2}\sum_{i=1}^{n} i^1 - \frac{2}{n}\sum_{i=1}^{4} 1\right]$$

$$A = \lim_{n\to\infty}\left[\frac{16}{n^4}\times\frac{n^4+2n^3+n^2}{4} - \frac{8}{n^3}\times\frac{2n^3+3n^2+n}{6} - \frac{8}{n^2}\times\frac{n^2+n}{2} - \frac{2}{n}\times\frac{n}{1}\right]$$

$$A = \lim_{n\to\infty}\left[\frac{16}{4}+\frac{32}{4n}+\frac{16}{4n^2}-\frac{16}{6}-\frac{24}{6n}-\frac{8}{6n^2}-\frac{8}{2}-\frac{8}{2n}-\frac{2}{1}\right]$$

$$A = \lim_{n\to\infty}\left[4-\frac{16}{6}-4-2\right] + \lim_{n\to\infty}\left[\frac{32}{4n}-\frac{24}{6n}-\frac{8}{2n}\right] + \lim_{n\to\infty}\left[\frac{16}{n^2}-\frac{8}{6n^2}\right]$$

$$A = -4\frac{2}{3}$$

(E) Exact Areas – Example #6

- Now we can confirm this visually using graphing software (I used WINPLOT):
- And we get the total area between the axis and the curve to be -4.666666666 from the software!