Lesson 46 – Area Under the Curve – Riemann Sums

Calculus - Santowski

(A) REVIEW

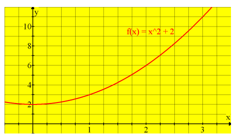
- We have looked at the process of anti-differentiation (given the derivative, can we find the “original” equation?)
- Then we introduced the indefinite integral ➔ which basically involved the same concept of finding an “original”equation since we could view the given equation as a derivative
- We introduced the integration symbol ➔ $\int$
- Now we will move onto a second type of integral ➔ the definite integral

(B) THE AREA PROBLEM

- to introduce the second kind of integral : Definite Integrals ➔ we will take a look at “the Area Problem” ➔ the area problem is to definite integrals what the tangent and rate of change problems are to derivatives.
- The area problem will give us one of the interpretations of a definite integral and it will lead us to the definition of the definite integral.

(B) THE AREA PROBLEM

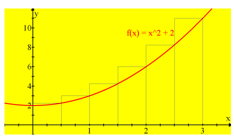
- Let’s work with a simple quadratic function, $f(x) = x^2 + 2$ and use a specific interval of [0,3]
- Now we wish to find the area under this curve



(C) THE AREA PROBLEM – AN EXAMPLE

- To estimate the area under the curve, we will divide the are into simple rectangles as we can easily find the area of rectangles ➔ $A = l \times w$
- Each rectangle will have a width of Δx which we calculate as $(b - a)/n$ where b represents the higher bound on the area (i.e. x = 3) and a represents the lower bound on the area (i.e. x = 0) and n represents the number of rectangles we want to construct
- The height of each rectangle is then simply calculated using the function equation
- Then the total area (as an estimate) is determined as we sum the areas of the numerous rectangles we have created under the curve
- $A_T = A_1 + A_2 + A_3 + ... + A_n$
- We can visualize the process on the next slide

(C) THE AREA PROBLEM – AN EXAMPLE



- We have chosen to draw 6 rectangles on the interval [0,3]
- $A_1 = ½ \times f(½) = 1.125$
- $A_2 = ½ \times f(1) = 1.5$
- $A_3 = ½ \times f(1½) = 2.125$
- $A_4 = ½ \times f(2) = 3$
- $A_5 = ½ \times f(2½) = 4.125$
- $A_6 = ½ \times f(3) = 5.5$
- $A_T = 17.375$ square units
- So our estimate is 17.375 which is obviously an overestimate

(C) THE AREA PROBLEM – AN EXAMPLE

- In our previous slide, we used 6 rectangles which were constructed using a “right end point” (realize that both the use of 6 rectangles and the right end point are arbitrary!) ➔ in an increasing function like $f(x) = x^2 + 2$ this creates an over-estimate of the area under the curve
- So let’s change from the right end point to the left end point and see what happens

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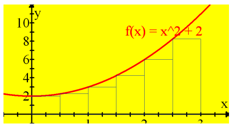
(C) THE AREA PROBLEM – AN EXAMPLE



- We have chosen to draw 6 rectangles on the interval [0,3]
- $A_1 = ½ \times f(0) = 1$
- $A_2 = ½ \times f(½) = 1.125$
- $A_3 = ½ \times f(1) = 1.5$
- $A_4 = ½ \times f(1½) = 2.125$
- $A_5 = ½ \times f(2) = 3$
- $A_6 = ½ \times f(2½) = 4.125$
- $A_T = 12.875$ square units
- So our estimate is 12.875 which is obviously an under-estimate

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(C) THE AREA PROBLEM – AN EXAMPLE

- So our “left end point” method (now called a left rectangular approximation method LRAM) gives us an underestimate (in this example)
- Our “right end point” method (now called a right rectangular approximation method RRAM) gives us an overestimate (in this example)
- We can adjust our strategy in a variety of ways ➔ one is by adjusting the “end point” ➔ why not simply use a “midpoint” in each interval and get a mix of over- and under-estimates? ➔ see next slide

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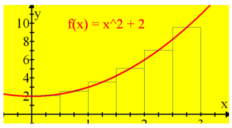
(C) THE AREA PROBLEM – AN EXAMPLE



- We have chosen to draw 6 rectangles on the interval [0,3]
- $A_1 = ½ \times f(¼) = 1.03125$
- $A_2 = ½ \times f(¾) = 1.28125$
- $A_3 = ½ \times f(1¼) = 1.78125$
- $A_4 = ½ \times f(1¾) = 2.53125$
- $A_5 = ½ \times f(2¼) = 3.53125$
- $A_6 = ½ \times f(2¾) = 4.78125$
- $A_T = 14.9375$ square units which is a more accurate estimate (15 is the exact answer)

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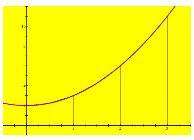
(C) THE AREA PROBLEM – AN EXAMPLE



- We have chosen to draw 6 trapezoids on the interval [0,3]
- $A_1 = ½ \times ½[f(0) + f(½)] = 1.0625$
- $A_2 = ½ \times ½[f(½) + f(1)] = 1.3125$
- $A_3 = ½ \times ½[f(1) + f(1½)] = 1.8125$
- $A_4 = ½ \times ½[f(1½) + f(2)] = 2.5625$
- $A_5 = ½ \times ½[f(0) + f(½)] = 3.5625$
- $A_6 = ½ \times ½[f(0) + f(½)] = 4.8125$
- $A_T = 15.125$ square units
- (15 is the exact answer)

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(D) THE AREA PROBLEM - CONCLUSION

- We have seen the following general formula used in the preceding examples:
- $A = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_i)\Delta x + \ldots. + f(x_n)\Delta x$ as we have created n rectangles
- Since this represents a sum, we can use summation notation to re-express this formula ➔

$$A = \sum_{i=1}^{n} f(x_i)\Delta x$$

- So this is the formula for our rectangular approximation method

(E) RIEMANN SUMS – INTERNET INTERACTIVE EXAMPLE

- Visual Calculus - Riemann Sums
- And some further worked examples showing both a graphic and algebraic representation:
- http://www.intmath.com/integration/riemann-sums.php
- http://mathworld.wolfram.com/RiemannSum.html

(F) THE AREA PROBLEM – EXAMPLES

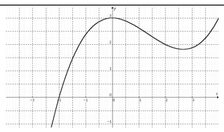
1. a) Approximate the area under the graph of $f(x)=\frac{1}{x}$ from $x=1$ to $x=5$ using the **right endpoints** of four subintervals of equal length. Sketch the graph and the rectangles. Is your estimate an underestimate or an overestimate?
   b) Repeat part a) using **left endpoints**.

2. Approximate the area under the graph of $f(x)=25-x^2$ from $x=0$ to $x=5$ using the **midpoints** of five subintervals of equal length. Sketch the graph and the rectangles.

3. a) Approximate the area under the graph of $f(x)=x^2+1$ from $x=-1$ to $x=2$ using the **right endpoints** of three subintervals of equal length. Sketch the graph and the rectangles.
   b) Improve your estimate by using six subintervals.
   c) Repeat parts a) and b) using **left endpoints**.
   d) Repeat parts a) and b) using **midpoints**.
   e) From your sketches in parts a), c), and d), which appears to be the best estimate?

(F) THE AREA PROBLEM – EXAMPLES

4.
   a) Approximate the area under the graph of the function shown to the right from $x=-2$ to $x=3$ using the **right endpoints** of five subintervals of equal length.
   b) Repeat part a) using **left endpoints**.
   c) Repeat part a) using **midpoints**.



(F) THE AREA PROBLEM – EXAMPLES

- Find the area between the curve $f(x) = x^3 - 5x^2 + 6x + 5$ and the x-axis on [0,4] using 5 intervals and using right- and left- and midpoint Riemann sums.
- Verify with technology.

(F) THE AREA PROBLEM – EXAMPLES

- Graph the function given below over the interval x = -1 to x = 2. Estimate the area under the graph of f using three approximating rectangles and taking the sample points to be:

$$f(x)=\frac{1}{1+x^2}$$

- a. Right endpoints
- b. Left endpoints
- c. Midpoints
- d. Trapezoids

(F) THE AREA PROBLEM – EXAMPLES

Consider the continuous function $f(x)$ such that $f(x)>0$ for $[0,1]$. Selected values of $f(x)$ are given in the table below. Use the table of values to approximate the area under $f(x)$ using the Riemann Sum indicated.

x	0	0.25	0.5	0.75	1.0
$f(x)$	1.0	0.8	1.3	1.1	1.6

a) Trapezoidal Approximation using 4 subintervals
b) Right Rectangular Approximation using 4 subintervals
c) Midpoint Rectangular Approximation using 2 subintervals

(F) THE AREA PROBLEM – EXAMPLES

- Let's put the area under the curve idea into a physics application: using a v-t graph, we can determine the distance traveled by the object
- During a three hour portion of a trip, Mr. S notices the speed of his car (rate of change of distance) and writes down the info on the following chart:

Time (hr)	0	1	2	3
Speed (m/h)	60	48	58	63

- Q: Use LHRS, RHRS & MPRS to estimate the total change in distance during this 3 hour portion of the trip

(G) THE AREA PROBLEM – FURTHER EXAMPLES

- So from our last example, an interesting point to note:
- The function/curve that we started with represented a **rate of change of distance** function, while the area under the curve represented a **total/accumulated change in distance**

(F) THE AREA PROBLEM – EXAMPLES

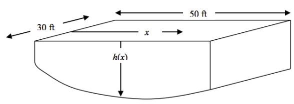
- Coal gas is produced at a gasworks. Pollutants are removed by screens which become less efficient as time goes on. Measurements are made every two months showing the rate at which pollutants escape..

Time (months)	0	2	4	6
Rate (tons/month)	5	8	13	20

- Find the amount of pollutants that escape using:
- a. lower estimate
- b. upper estimate

(F) THE AREA PROBLEM – EXAMPLES

A swimming pool with a rectangular surface is 30 ft wide and 50 ft long. The volume of the pool is shaped as a prism (see drawing.) The table below shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other.



a) Estimate the lateral area of the pool using a Riemann sum with the **midpoints** of five subintervals of equal length.

b) Use this information to calculate the volume of water in the pool. *(Hint: remember that the volume of a prism is the area of its base times the distance between the bases.)*

Data for the swimming pool											
Position (ft): x	0	5	10	15	20	25	30	35	40	45	50
Depth (ft): $h(x)$	6.0	8.2	9.1	9.9	10.5	11.0	11.5	11.9	12.3	12.7	13.0

(I) INTERNET LINKS

- Calculus I (Math 2413) - Integrals - Area Problem from Paul Dawkins
- Integration Concepts from Visual Calculus