Lesson 43 - Related Rates

Calculus - Santowski

Lesson Objectives

- 1. Explain what the notation d/dt means
- 2. Given a situation in which several quantities vary, predict the rate at which one of the quantities is changing when you know the other related rates

Fast Five

- 1. Determine dy/dt for $y(t) = t^2$
- 2. Determine dy/dt for $y(x) = x^2$ where x(t)
- 3. Give a practical meaning for dy/dx
- 4. Give a practical meaning for dV/dt
- 5. Give a practical meaning for dV/dr
- 6. Determine dV/dr if $V = 1/3\pi r^2 h$
- 7. Determine dV/dh if $V = 1/3\pi r^2 h$
- 8. Determine dV/dt if $V = 1/3\pi r^2 h$

(A) Review

- Recall that the meaning of dy/dx is a change of *y* as we change *x*
- Recall that a rate can also be understood as a change of some quantity with respect to time ➔ As a derivative, we would write this as *dX / dt*
- So therefore d*V* /d*t* would mean ➔ the rate of change of the volume as time changes
- Likewise d*A*/ d*r* ➔ the rate of change of the area as we make changes in the radius
- So also consider:
  - d*V* / d*r* ➔
  - d*h* / d*r* ➔
  - d*h* / d*t* ➔
  - d*r* / d*t* ➔

(B) Related Rates and Circles

- example 1
- A pebble is dropped into a pond and the ripples form concentric circles. The radius of the outermost circle increases at a constant rate of 10 cm/s. Determine the rate at which the area of the disturbed water is changing when the radius is 50 cm.

(B) Related Rates and Circles

- ex 1. A pebble is dropped into a pond and the ripples form concentric circles. The radius of the outermost circle increases at a constant rate of 10 cm/s. Determine the rate at which the area of the disturbed water is changing when the radius is 50 cm.
- So, as the radius changes with time, so does the area ➔ dr/dt is related to dA/dt ➔ how?
- Recall the area formula ➔ $A = \pi r^2$ ➔ $A(t) = \pi(r(t))^2$
- Then use implicit differentiation as we now differentiate with respect to time ➔ $d/dt\,(A(t)) = d/dt\,(\pi(r(t))^2) = 2\pi r \times dr/dt$
- So now we know the relationship between dA/dt and dr/dt
- Then if we knew dr/dt, we could find dA/dt
- It is given that dr/dt is 10 cm/sec, then $dA/dt = 2 \times \pi \times (50\text{cm}) \times 10$ cm/sec
- $dA/dt = 1000\pi$ cm^2/sec or 3141 square cm per second

(C) Related Rates and 3D Volumes

- Example 2.
- A water tank has a shape of an inverted circular cone with a base radius of 2 m and a height of 4 m. If water is being pumped in a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 meters deep.

(C) Related Rates and 3D Volumes

- Ex 2. A water tank has a shape of an inverted circular cone with a base radius of 2 m and a height of 4 m. If water is being pumped in a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 meters deep.
- So in the context of this conical container filling, we see that the rate of change of the volume is related to 2 different rates ➔ the rate of change of the height and the rate of change of the radius
- Likewise, the rate of change of the height is related to the rate of change of the radius and the rate of change of time.
- So as we drain the conical container, several things change ➔ V,r,h,t and how they change is related

(C) Related Rates and 3D Volumes

- Having the formula $V = 1/3\pi r^2 h$ (or recall that $V(t) = 1/3\pi(r(t))^2 h(t)$) and the differentiation $dV/dt = d/dt(1/3\pi r^2 h)$, we can now take the derivative
- $dV/dt = 1/3\pi \times d/dt\,[(r^2 \times h)]$
- $dV/dt = 1/3\pi \times [\,(2r \times dr/dt \times h) + (dh/dt \times r^2)]$
- So as we suspected initially, the rate at which the volume in a cone changes is related to the rate at which the radius changes and the rate at which the height changes.
- If we know these 2 rates (dr/dt and dh/dt) we can solve the problem
- But if we do not know the 2 rates, we need some other relationship to help us out.
- In the case of a cone ➔ we usually know the relationship between the radius and the height and can express one in terms of the other

(C) Related Rates and 3D Volumes

- In a right angled cone (radius is perpendicular to the height) the ratio of height to radius is always constant
- In this case, $h/r = 4/2$ ➔ so $r = ½h$
- So our formula $V = 1/3\pi r^2 h$ becomes $V = 1/3\pi(½h)^2 h = 1/12\pi h^3$
- Now we can differentiate again
- $dV/dt = d/dt\,(1/12\pi h^3)$
- $dV/dt = 1/12\pi \times 3h^2 \times dh/dt$
- $2\ m^3/min = 1/12\pi \times 3(3)^2 \times dh/dt$
- $dh/dt = 2 \div (27/12\pi)$
- $dh/dt = 8\pi/9$ m/min

(D) Pythagorean Relationships

- Example 3
- A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the foot of the wall?

(D) Pythagorean Relationships

- A ladder 10 meters long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 m from the foot of the wall?
- If we set up a diagram, we create a right triangle, where the ladder represents the hypotenuse and then realize that the quantities that change with time are the distance of the foot of the ladder along the floor (*x*) and the distance from the top of the ladder to the floor (*y*)
- So we let x represent this distance of the foot of the ladder and y represents the distance of the top of the ladder to the floor
- So our mathematical relationship is $x^2 + y^2 = 10^2$

(D) Pythagorean Relationships

- So we have $x^2 + y^2 = 10^2$ and we simply differentiate wrt time
- Then $d/dt\,(x^2 + y^2) = d/dt\,(10^2)$
- $2x \times dx/dt + 2y \times dy/dt = 0$
- $x \times dx/dt = - y \times dy/dt$
- Then $dy/dt = x \times dx/dt \div - y$
- So $dy/dt = (6 \text{ m}) \times (1 \text{ m/s}) \div -(8) =$
- $dy/dt = -3/4$ m/sec

- So the top of the ladder is coming down at a rate of 0.75 m/sec when the foot of the ladder is 6 m from the wall

(E) Related Rates and Angles

- Example 4

- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?

(E) Related Rates and Angles

- You walk along a straight path at a speed of 4 m/s. A search light is located on the ground, a perpendicular distance of 20 m from your path. The light stays focused on you. At what rate does the search light rotate when you are 15 meters from the point on the path closest to the search light?

- So we need a relationship between the angle, the 20 meters and your distance along the path ➔ use the primary trig ratios to set this up ➔ the angle is that between the perpendicular (measuring 20 meters) and the path of the opposite side ➔ opposite and adjacent are related by the tangent ratio

- So $\tan(\theta) = x/20$ or $x = 20\tan(\theta)$
- Differentiating ➔ $d/dt\,(x) = d/dt\,(20\tan(\theta))$
- Thus $dx/dt = 20 \times \sec^2(\theta) \times d\theta/dt = 4$ m/s
- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$

(E) Related Rates and Angles

- Then $d\theta/dt = 4 \div 20\sec^2(\theta)$

- To find $\sec^2(\theta)$, the measures in our triangle at the instant in question are the 20m as the perpendicular distance, 15m as the distance from the perpendicular, and then the hypotenuse as 25m ➔ so $\sec^2(\theta)$, $= (25/15)^2 = 25/16$

- Then $d\theta/dt = 4 \div (20 \times 25 \div 16) = 0.128$ rad/sec

- So the search light is rotating at 0.128 rad/sec

(F) Summary

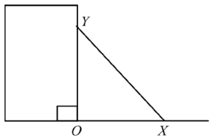
- In this section we've seen four related rates problems. They all work in essentially the same way. The main difference between them was coming up with the relationship between the known and unknown quantities. This is often the hardest part of the problem.

- The best way to come up with the relationship is to sketch a diagram that shows the situation. This often seems like a silly step, but can make all the difference in whether we can find the relationship or not.

AP Calculus

Related Rates Free Response

1982 AB4



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

Video

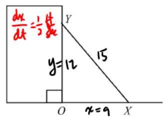
1982 AB 4

$\frac{dy}{dt} = ??$ when $x = 9$

$x^2 + y^2 = 15^2$

$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$

$9(1/2) + 12\, dy/dt = 0$



A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building. $\frac{dy}{dt} = -\frac{9}{24} = -\frac{3}{8}$ ft/sec.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

$A = \frac{1}{2}bh = \frac{1}{2}xy$

$\frac{dA}{dt} = \frac{1}{2}x\frac{dy}{dt} + y\frac{1}{2}\frac{dx}{dt}$

$= \frac{1}{2}(9)\left(-\frac{3}{8}\right) + (12)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{27}{16} + \frac{\cancel{12}^{3}}{4} = \frac{48}{16} - \frac{27}{16} = \frac{21}{16}$ ft²/sec

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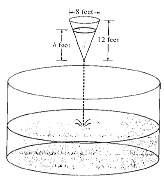
1995 AB5/BC3



As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h.

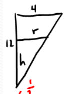
(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.

Video

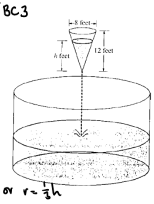
1995 AB5/BC3

$\frac{dh}{dt} = h - 12$



$\frac{\cancel{4}^{1}}{\cancel{12}_{3}} = \frac{r}{h}$

$3r = h$ or $r = \frac{1}{3}h$



$V = \frac{1}{3}\pi r^2 h$

$= \frac{\pi}{3}\left(\frac{1}{3}h\right)^2 h$

$= \frac{\pi}{3}\left(\frac{1}{9}h^2\right)h$

$= \boxed{\frac{\pi}{27}h^3}$

$\frac{dV}{dt} = \frac{3\pi}{27}h^2\frac{dh}{dt}$

$= \frac{\cancel{3\pi}}{\cancel{27}}(\cancel{3})^2(-9)$

$= \boxed{-9\pi \text{ ft}^3/\text{min}}$

$V = Bh$

$V = 400\pi y$

$\frac{dV}{dt} = 400\pi\frac{dy}{dt}$

$9\pi = 400\pi\boxed{\frac{dy}{dt}}$

$\frac{dy}{dt} = \boxed{\frac{9}{400} \text{ ft}^3/\text{min}}$

As shown in the figure above, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h-12)$ feet per minute. (The volume V of a cone with radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

(a) Write an expression for the volume of water in the conical tank as a function of h. $V = (\pi/27)h^3$

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure. -9π ft³/min

(c) Let y be the depth, in feet, of the water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure. $9/400$ ft³/min

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1990 AB4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second.

(Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.)

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius?

Video

1990 AB 4

The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second.

(Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.) $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} = 4\pi(100)\left(\frac{4}{100}\right) = 16\pi$ cm³/sec.

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume? 16π cm³/sec.

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere? $.24\pi$ cm²/sec.

(c) At the time when the volume and the radius of the sphere are increasing at the same numerical rate, what is the radius? $\frac{1}{2\sqrt{\pi}}$

$V = \frac{4}{3}\cancel{\pi}r^3 = 36\cancel{\pi}$

$r^3 = 27$

$r = 3$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(3)\left(\frac{4}{100}\right) = \frac{24\pi}{100} = .24\pi\ \frac{\text{cm}^2}{\text{sec.}}$

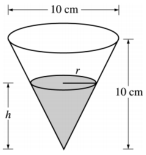
when $\frac{dV}{dt} = \frac{dr}{dt}$, $r = ??$

$4\pi r^2\frac{dr}{dt} = \frac{dr}{dt}$

$4\pi r^2 = 1$

$r^2 = \frac{1}{4\pi}$ $\quad r = \frac{1}{\sqrt{4\pi}} = \frac{1}{2\sqrt{\pi}}$

2002 AP® CALCULUS AB FREE-RESPONSE QUESTIONS



5. A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr.

(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.)

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

Video

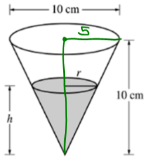
2002 AB 5

$\frac{dV}{dt} = k\pi r^2$

$-\frac{75\pi}{40} = k\frac{25\pi}{4}$

$k = \frac{-75\pi}{40}\left(\frac{4}{25\pi}\right)$

$= -\frac{3}{10}$



$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12}h^3$

$\frac{dV}{dt} = \frac{3\pi}{12}h^2\frac{dh}{dt}$

$= \frac{\pi}{4}(25)\left(-\frac{3}{10}\right) = \boxed{\frac{-75\pi}{40}\frac{cm^3}{hr}}$

A container has the shape of an open right circular cone, as shown in the figure above. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/hr. $\frac{dh}{dt} = -\frac{3}{10}$ cm/hr. $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}\left(\frac{5}{2}\right)^2 5 = \boxed{\frac{125\pi}{12} cm^3}$

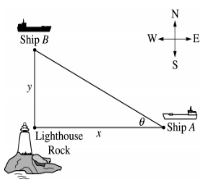
(Note: The volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h$.) $\frac{5}{10} = \frac{r}{h}$ $10r = 5h$ $r = \frac{1}{2}h$

(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure. $\frac{125}{12}$ cm^3

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure. $\frac{dV}{dt} = ??$ when $h=5$ $-75\pi/40$ $cm^3/hr.$

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

2002 AP® CALCULUS AB FREE-RESPONSE QUESTIONS (Form B)



6. Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km.

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km.

Video

2002 AB 6 Form B (No Calculator)

$\tan\theta = y/x$

$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$

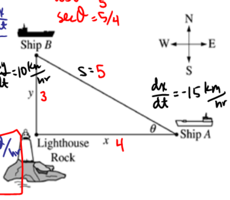
$\cos\theta = \frac{4}{5}$

$\sec\theta = 5/4$

$\frac{25}{16}\frac{d\theta}{dt} = \frac{4(10) - 3(-15)}{16}$ $\frac{dy}{dt} = 10\frac{km}{hr}$

$\frac{25}{16}\boxed{\frac{d\theta}{dt}} = \frac{85}{16}$

$\frac{d\theta}{dt} = \frac{85}{16}\cdot\frac{16}{25} = \frac{85}{25} = \boxed{\frac{17}{5}\ rad/hr}$



$\frac{dx}{dt} = -15\frac{km}{hr}$

$x^2 + y^2 = s^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

$x\frac{dx}{dt} + y\frac{dy}{dt} = s\boxed{\frac{ds}{dt}}$

$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{s}$

$= \frac{4(-15) + 3(10)}{5}$

$= \frac{-60+30}{5} = \boxed{-6\frac{km}{hr}}$

Ship A is traveling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is traveling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure above.

(a) Find the distance, in kilometers, between Ship A and Ship B when $x = 4$ km and $y = 3$ km. 5km

(b) Find the rate of change, in km/hr, of the distance between the two ships when $x = 4$ km and $y = 3$ km. $-6\frac{km}{hr}$

(c) Let θ be the angle shown in the figure. Find the rate of change of θ, in radians per hour, when $x = 4$ km and $y = 3$ km. $17/5$ rad/hr.