Lesson 42 – Rates of Change (Applications of Derivatives) - Motion

Math HL1 - Santowski

Lesson Objectives

- Apply derivatives to work with rates of change in various contexts:
- (A) Kinematics
- (B) Economics
- (C) Other Natural Science contexts

(A) Kinematics (CI)

- Ex 1 – If a stone is dropped from a cliff that is 122.5 meters high, then its height, in meters, after t seconds is $h(t) = 122.5 - 4.9t^2$.
- (a) Find the velocity of the stone at $t = 1.0$ sec and at $t = 2.0$ seconds.
- (b) When will the stone hit the ground?
- (c) With what velocity will the stone hit the ground?

(A) Kinematics (CA)

- Ex 2 – A projectile is launched upwards and experiences resistance such that its height above the ground after a time of t seconds is modeled by $h(t) = 220(1 - e^{-0.2t}) - 20t$
- (a) Find the initial velocity of the projectile.
- (b) What is the maximum height of the projectile?
- (c) With what velocity will the stone hit the ground?

Rectilinear Motion

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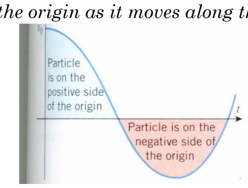
- *The position function of the particle is s(t) and we call the graph of s versus t the position vs. time curve.*
- *The change in the position of the particle is called the displacement of the particle. The displacement describes where it is compared to where it started.*

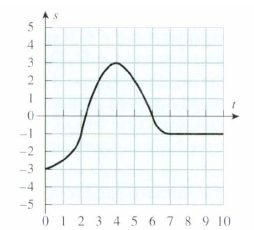
Rectilinear Motion

- *The figure below is a typical position vs. time curve for a particle in rectilinear motion. We can tell from the graph that the coordinate of the particle at t = 0 is s_0, and we can tell from the sign of s when the particle is on the negative or the positive side of the origin as it moves along the coordinate line.*



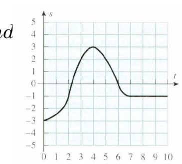
Example 1

- *The figure below shows the position vs. time curve for a particle moving along an s-axis. In words, describe how the position of the particle is changing with time.*



Example 1

- *The figure below shows the position vs. time curve for a particle moving along an s-axis. In words, describe how the position of the particle is changing with time.*
- *At t = 0, s(t) = -3. It moves in a positive direction until t = 4 and s(t) = 3. Then, it turns around and travels in the negative direction until t = 7 and s(t) = -1. The particle is stopped after that.*



Velocity and Speed

- *The rate of change of your position is based on your velocity. The rate of change is the first derivative. This leads us to:*

$$v(t) = s'(t) = \frac{ds}{dt}$$

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- *Remember, velocity has direction attached to it. If velocity is positive, the particle is moving to the right or up. If velocity is negative, the particle is moving to the left or down.*

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- *Remember, velocity has direction attached to it. If velocity is positive, the particle is moving to the right or up. If velocity is negative, the particle is moving to the left or down.*
- *Speed is just how fast you are going regardless of direction. This leads us to:*

$$|v(t)| = |s'(t)| = speed$$

Example 2

- *Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along the s-axis, where s is in meters and t is in seconds. Find the velocity and speed functions, and show the graphs of position, velocity, and speed versus time.*

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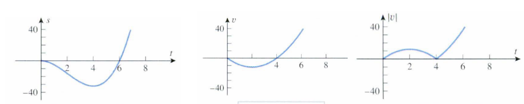
$$v(t) = \frac{ds}{dt} = 3t^2 - 12t \qquad speed = |v(t)| = |3t^2 - 12t|$$

Example 2

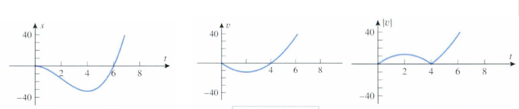
- *The graphs below provide a wealth of visual information about the motion of the particle.*
- *Describe what information the three graphs present to us about the motion of the particle*



Position versus time   Velocity versus time   Speed versus time

Example 2

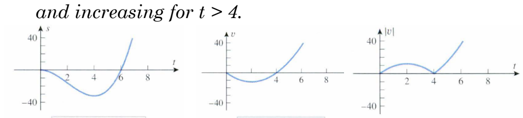
- *The graphs below provide a wealth of visual information about the motion of the particle. For example, the position vs. time curve tells us that the particle is on the negative side of the origin for $0 < t < 6$, is on the positive side of the origin for $t > 6$ and is at the origin at times $t = 0$ and $t = 6$.*



Position versus time   Velocity versus time   Speed versus time

Example 2

- *The velocity vs. time curve tells us that the particle is moving in the negative direction if $0 < t < 4$, and is moving in the positive direction if $t > 4$ and is stopped at times $t = 0$ and $t = 4$ (the velocity is zero at these times). The speed vs. time curve tells us that the speed of the particle is increasing for $0 < t < 2$, decreasing for $2 < t < 4$ and increasing for $t > 4$.*



Position versus time   Velocity versus time   Speed versus time

Acceleration

- *The rate at which the instantaneous velocity of a particle changes with time is called instantaneous acceleration. We define this as:*

$$a(t) = v'(t) = s''(t) = \frac{dv}{dt}$$

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- *We now know that the first derivative of position is velocity and the second derivative of position is acceleration.*

Example 3

- *Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis where s is in meters and t is in seconds. Find the acceleration function a(t) and show that graph of acceleration vs. time.*

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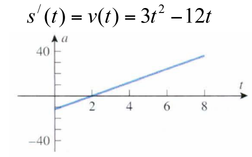
- *Let $s(t) = t^3 - 6t^2$ be the position function of a particle moving along an s-axis where s is in meters and t is in seconds. Find the acceleration function a(t) and show that graph of acceleration vs. time.*

$$s'(t) = v(t) = 3t^2 - 12t$$



Acceleration versus time

$$s''(t) = a(t) = 6t - 12$$

$$6(t-2)$$

Speeding Up and Slowing Down

- *We will say that a particle in rectilinear motion is speeding up when its speed is increasing and slowing down when its speed is decreasing. In everyday language an object that is speeding up is said to be "accelerating" and an object that is slowing down is said to be "decelerating."*

- *Whether a particle is speeding up or slowing down is determined by both the velocity and acceleration.*

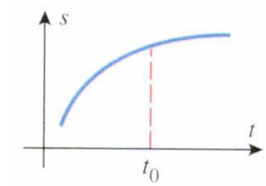
The Sign of Acceleration

- *A particle in rectilinear motion is speeding up when its velocity and acceleration have the same sign and slowing down when they have opposite signs.*

Analyzing the Position vs. Time Curve

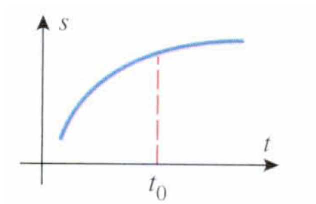
- *If $s(t) > 0$, the particle is on the positive side of the s-axis and if $s(t) < 0$, the particle is on the negative side of the s-axis.*
- *The slope of the curve at any time is equal to the instantaneous velocity at that time.*
- *Where the curve has positive slope, the velocity is positive and the particle is moving in the positive direction.*
- *Where the curve has negative slope, the velocity is negative and the particle is moving in the negative direction.*
- *Where the curve has slope zero, the velocity is zero and the particle is momentarily stopped.*

Position vs. Time Curve



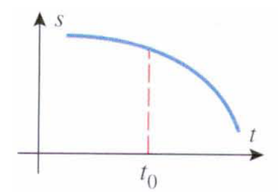
- *From this position vs. time curve determine if at t_0:*
  - *(a) The particle is on the positive or negative side of the origin.*
  - *(b) The direction the particle is moving.*
  - *(c) The particle is speeding up or slowing down.*

Position vs. Time Curve



- *The particle is on the positive side of origin*
- *1st derivative is positive, so the particle is moving in the positive direction*
- *2nd derivative is negative (concave down), so the particle is slowing down.*

Position vs. Time Curve



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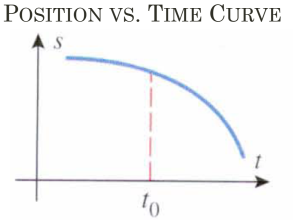
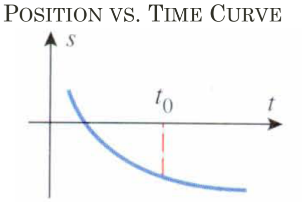


- *The particle is on the positive side of origin*
- *1st derivative is negative, so the particle is moving in the negative direction*
- *2nd derivative is negative, so the particle is speeding up.*

Position vs. Time Curve



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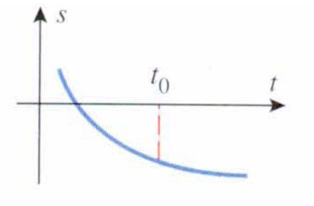
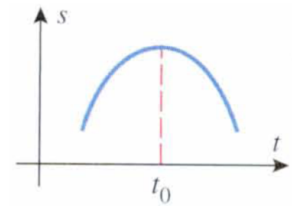


- *The particle is on the negative side of origin*
- *1st derivative is negative, so the particle is moving in the negative direction*
- *2nd derivative is positive, so the particle is slowing down.*

Position vs. Time Curve



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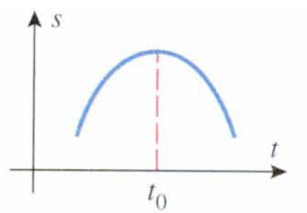

- *The particle is on the positive side of origin*
- *The particle is momentarily stopped.*
- *The velocity is decreasing.*

Example 6

- *Suppose that the position function of a particle moving on a coordinate line is given by* $s(t) = 2t^3 - 21t^2 + 60t + 3$
- *Analyze the motion of the particle.*

Example 6

$$s(t) = 2t^3 - 21t^2 + 60t + 3$$

$$v(t) = 6t^2 - 42t + 60 = 6(t-2)(t-5)$$

right *left* *right*

|___+___ |____-____ |___+___

0 *2* *5*

stop *stop*

$$a(t) = 12t - 42 = 12(t - 3.5)$$

|_____-______ |____+______

0 *3.5*

Example 6

right *left* *right*

|___+___ |____-____ |___+___

0 *2* *5*

|__-____ |__-_ |__+__ |__+____

0 *2* *3.5* *5*

- *From 0-2, moving to the right and slowing down.*
- *From 2-3.5, left, speeding up.*
- *From 3.5-5, left, slowing down.*
- *From 5 on, right and speeding up.*

EXAMPLE 6

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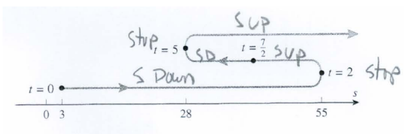
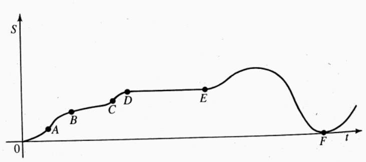


(A) KINEMATICS

- Ex 2 – The position of a particle moving on a line is modeled by $s(t) = 2t^3 - 21t^2 + 60t$, $t > 0$, where t is measured in seconds and s in meters.
- (a) What is the velocity after 3 s and after 6 s?
- (b) When is the particle at rest?
- (c) When is the particle moving in the positive direction?
- (d) Find the total distance traveled by the particle during the first 6 seconds

(A) KINEMATICS

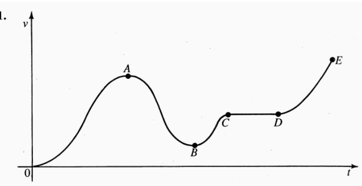
1. The graph shows the position function of a car.



(a) What was the initial velocity of the car?
(b) Was the car going faster at B or at C?
(c) Was the car slowing down or speeding up at A, B, and C?
(d) What happened between D and E?
(e) What happened at F?

(A) KINEMATICS

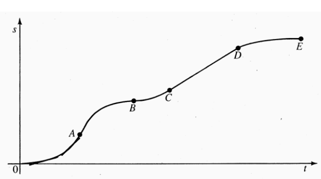
1.



The graph of a velocity function is shown. State whether the acceleration is positive, zero, or negative
(a) from O to A, (b) from A to B,
(c) from B to C, (d) from C to D,
(e) from D to E.

(A) KINEMATICS

2.



The graph of a position function is shown.
(a) For the part of the graph from O to A, use slopes of tangents to decide whether the velocity is increasing or decreasing. Is the acceleration positive or negative?
(b) State whether the acceleration is positive, zero, or negative
(i) from A to B (ii) from B to C
(iii) from C to D (iv) from D to E

(A) TERMINOLOGY AND BACKGROUND

- Cost Function refers to how much it costs to produce x units of a commodity and is given by the equation y = C(x) and its units are usually in $
- An average cost function is determined by dividing the total production cost by the total production level: AC(x)=c(x)=C(x)/x and its units are $ per item
- The demand function represents the price per unit, p(x),that a company can charge if it sells x units. The price function units are obviously $. (the inverse of the demand function, x(p), can easily be understood as the number of units that can be sold, x, if the price is p)
- The revenue function, R(x), is simply determined by the product of the price charged multiplied by the number of items sold, in other words, R(x) = px
- The profit function, P(x), is simply the difference between the revenue function and the cost function, P(x) = R(x) – C(x)
- A break-even point is the production level at which the revenue generated is equal to the cost of production, in other words, when the profit is 0.

(B) Derivatives of these Functions

- The derivative of the cost function, dC/dx, is called the marginal cost and represents the rate of change of costs as the production level changes. Its units are $/item. Usually, we interpret dC/dx as the change in cost when producing one more item, given a specific production level.
- The derivative of the revenue function, dR/dx, is called the marginal revenue and represents the rate of change of revenues as the sales level changes. Its units are $/item. Usually, we interpret dR/dx as the change in revenue when selling one more item, given a specific sales level.
- The derivative of the profit function, dP/dx, is called the marginal profit and represents the rate of change of revenues as the sales level changes. Its units are $/item. Usually, we interpret dP/dx as the change in revenue when selling one more item, given a specific sales level.

Example 1

- General Mills is a large producer of flour. The production manager estimates that the daily cost in producing x bags of flour is given by the equation: $C(x) = 140{,}000 + 0.43x + 0.000\,001x^2$
- a. Find the average cost of producing 1000 bags
- b. Find the marginal cost at a production level of 1000 bags
- c. Find the actual cost of producing the 1001st bag.
- d. Find the average cost and the marginal cost of producing 100,000 bags.
- e. At what production level will the average cost be the smallest, and what is this average cost?

Example 2

- Bernie's Burger Barn (BBB) has determine that the yearly demand function for their hamburgers is given the equation $p(x) = (800{,}000 - x)/200{,}000$.
- a. Graph the demand function.
- b. Find the revenue generated at a sales level of 300,000 burgers.
- c. Find the marginal revenue when the sales level is 300,000 burgers.

Example 3

- The accountant at BBB has estimated that the cost function is $C(x) = 125{,}000 + 0.43x$. Using the demand function from Ex 2, determine:
- a. The profit when the sales level is 300,000
- b. The marginal profit when the sales level is 300,000
- c. What sales level will maximize profits?

Example 4

- A store has been selling 200 IPODS a week at $350 each. A market survey indicates that for each $10 rebate offered to the buyers, the number of sets sold will increase by 20 per week.
- a. Determine the demand function and the revenue functions.
- b. How large a rebate should be offered to maximize the revenue?