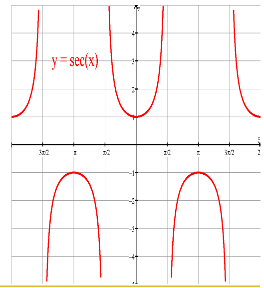
Lesson 40 – Derivatives of Secondary Trig Functions & Inverse Trig Functions

IB Math HL - Santowski

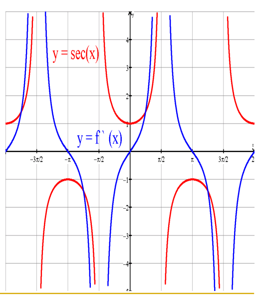
(A) Derivatives of f(x) = sec(x) - Graphically

- For y = sec(x) on (-2π,2π)
- Fcn is con up on (-2π,-3π/2), (-π/2, π/2), (3π/2,2π)
- Fcn is con down elsewhere
- Fcn has max at -π, π
- Fcn has min at -2π,0,2π
- Fcn increases on (-2π,-3π/2), (-3π/2,-π), (0,π/2), (π/2,π)
- Fcn decreases elsewhere
- Fcn has asymptotes at ±3π/2, ±π/2



(A) Derivatives of f(x) = sec(x) - Graphically

- For y = sec(x) on (-2π,2π)
- Fcn is con up on (-2π,-3π/2), (-π/2, π/2), (3π/2,2π) → ∴ f ` increases here
- Fcn is con down elsewhere → ∴ f ` decreases here
- Fcn has max at -π, π → roots on f `
- Fcn has min at -2π,0,2π → roots on f `
- Fcn increases on (-2π,-3π/2), (-3π/2,-π), (0,π/2), (π/2,π) → f ` is positive
- Fcn decreases elsewhere → f ` is negative
- Fcn has asymptotes at ±3π/2, ±π/2 → f ` has asymptotes



(B) Derivative of f(x) = sec(x) - Algebraically

- We will use the fact that sec(x) = 1/cos(x) to find the derivative of sec(x)

$$\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)$$
$$\frac{d}{dx}(\sec(x)) = \frac{\frac{d}{dx}(1)\times\cos(x) - \frac{d}{dx}(\cos(x))\times 1}{\cos^2 x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{(0)\times\cos x + \sin x}{\cos^2 x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{\sin x}{\cos^2 x} = \frac{1\times\sin x}{\cos x\times\cos x}$$
$$\frac{d}{dx}(\sec(x)) = \frac{1}{\cos x}\times\frac{\sin x}{\cos x}$$
$$\frac{d}{dx}(\sec(x)) = \sec x\times\tan x$$

(C) Derivatives of f(x) = csc(x) and f(x) = cot(x)

- We can run through a similar curve analysis and derivative calculations to find the derivatives of the cosecant and cotangent functions as well
- The derivatives turn out to be as follows:
- d/dx csc(x) = -csc(x) cot(x)
- d/dx cot(x) = -csc^2(x)

(D) Summary of Trig Derivatives

- primary trig fcns:

$$\frac{d}{dx}\sin(x) = \cos(x)$$
$$\frac{d}{dx}\cos(x) = -\sin(x)$$
$$\frac{d}{dx}\tan(x) = \sec^2(x)$$

- secondary trig fcns:

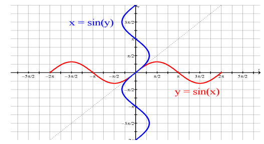
$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$$
$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$$
$$\frac{d}{dx}\cot(x) = -\csc^2(x)$$

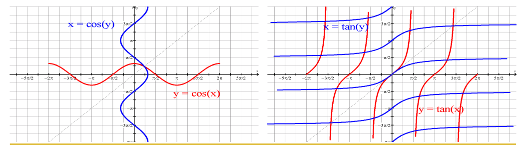
(E) Examples

- (i) Differentiate $f(x) = \dfrac{1}{1+\tan(x)}$
- (ii) Differentiate $h(x) = 2\csc^2(3x^2)$
- (iii) find dy/dx if tan(y) = x^2
- (iv) find the slope of the tangent line to y = tan(csc(x)) when sin(x) = $1/\pi$ on the interval $(0,\pi/2)$
- (v) Find the intervals of concavity of y = sec(x) + tan(x)

(A) Graphs of Inverse Trig Functions

- The graphs of the inverse trig functions are as follows:



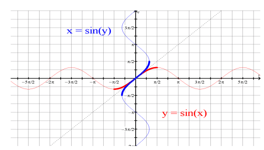


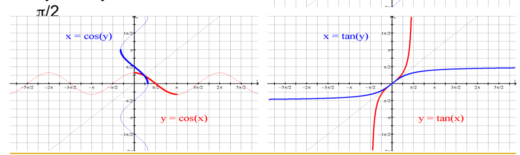
(B) Inverse Trig as Functions – Restrictions

- From the graphs previously shown, the inverse trig “relations” are not functions since the domain elements do not “match” the range elements i.e. ➔ not one-to-one
- So we need to make domain restrictions in the original function such that when we “invert”, our inverse does turn out to be a function
- What domain restrictions shall we make??

(B) Inverse Trig as Functions – Restrictions

- For y = sin(x) ➔between a max and min ($-\pi/2$ and $\pi/2$)
- For y = cos(x) ➔ between a max and min (0 and π)
- For y = tan(x) ➔ use one cycle, say between $-\pi/2$ and $\pi/2$





(C) Derivative of f(x) = $\sin^{-1}(x)$ on $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$

- If y = sin(x), then to make the inverse, x = sin(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin(y))$$
$$1 = \frac{d}{dy}(\sin(y)) \times \frac{dy}{dx}$$
$$1 = \cos(y) \times \frac{dy}{dx}$$
$$\frac{1}{\cos(y)} = \frac{dy}{dx}$$

- But can we make a substitution for cos(y)??

$$\sin^2 y + \cos^2 y = 1$$
$$\cos y = \sqrt{1-\sin^2 y} \Leftrightarrow x = \sin y$$
$$\cos y = \sqrt{1-x^2}$$
$$\Updownarrow$$
$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\cos y}$$
$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

(D) Derivative of f(x) = $\cos^{-1}(x)$ on $(0,\pi)$

- If y = cos(x), then to make the inverse, x = cos(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos(y))$$
$$1 = \frac{d}{dy}(\cos(y)) \times \frac{dy}{dx}$$
$$1 = -\sin(y) \times \frac{dy}{dx}$$
$$-\frac{1}{\sin(y)} = \frac{dy}{dx}$$

- But can we make a substitution for sin(y)??
-

$$\sin^2 y + \cos^2 y = 1$$
$$\sin y = \sqrt{1-\cos^2 y} \Leftrightarrow x = \cos y$$
$$\sin y = \sqrt{1-x^2}$$
$$\Updownarrow$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sin y}$$
$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = -\frac{1}{\sqrt{1-x^2}}$$

(E) Derivative of f(x) = $\tan^{-1}(x)$ on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

- If y = tan(x), then to make the inverse, x = tan(y) and we can use implicit differentiation to find dy/dx

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan(y))$$
$$1 = \frac{d}{dy}(\tan(y)) \times \frac{dy}{dx}$$
$$1 = \sec^2(y) \times \frac{dy}{dx}$$
$$\frac{1}{\sec^2(y)} = \frac{dy}{dx}$$

- But can we make a substitution for $\sec^2(y)$??

$$\sec^2 y = 1 + \tan^2 y$$
$$\sec^2 y = 1 + (\tan y)^2 \Leftrightarrow x = \tan y$$
$$\sec^2 y = 1 + x^2$$
$$\Updownarrow$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{\sec^2 y}$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

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(F) Summary of Trig Inverse Derivatives

- The three derivatives of the inverse of the trig. primary functions are:

$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

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(H) Examples

- Problems and Solutions to Differentiation of Inverse Trigonometric Functions from UC Davis

- Differentiate y = $\sin^{-1}(1-x^2)$
- Differentiate $f(x) = x\tan^{-1}\sqrt{x}$
- Differentiate y = $\cos^{-1}(\sin(x))$
- If y = $\tan^{-1}(x/y)$, find dy/dx

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(F) Homework

- Stewart, 1989, Chap 7.3, p319, Q1-4,6,7
- Stewart, 1989, Chap 7.6, p339, Q1-4

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