Lesson 39 - Derivatives of Primary Trigonometric Functions

IB Math HL - Santowski

Fast Five

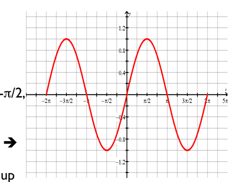
- 1. State the value of $\sin(\pi/4)$, $\tan(\pi/6)$, $\cos(\pi/3)$, $\sin(\pi/2)$, $\cos(3\pi/2)$
- 2. Solve the equation $\sin(2x) - 1 = 0$
- 3. Expand $\sin(x + h)$
- 4. State the value of $\sin^{-1}(0.5)$, $\cos^{-1}(\sqrt{3}/2)$

Lesson Objectives

- (1) Work with basic strategies for developing new knowledge in Mathematics ➔ (a) graphical, (b) technology, (c) algebraic
- (2) Introduce & work with fundamental trig limits
- (3) Determine the derivative of trigonometric functions
- (4) Apply & work with the derivatives of the trig functions

(A) Derivative of the Sine Function - Graphically

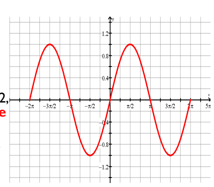
- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and $-3\pi/2$ ➔ derivative must have ……..? ?
- (ii) we see a minimum at $-\pi/2$ and $3\pi/2$ ➔ derivative must have ……..? ?
- (iii) we see intervals of increase on $(-2\pi,-3\pi/2)$, $(-\pi/2,\pi/2)$, $(3\pi/2,2\pi)$ ➔ derivative must …….?
- (iv) the opposite is true of intervals of decrease
- (v) intervals of concave up are $(-\pi,0)$ and $(\pi,2\pi)$ ➔ so derivative must …….? ?
- (vi) the opposite is true for intervals of concave up



- So the derivative function must look like ➔ ??

(A) Derivative of the Sine Function - Graphically

- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and $-3\pi/2$ ➔ derivative must have **ZEROES here**
- (ii) we see a minimum at $-\pi/2$ and $3\pi/2$ ➔ derivative must have **ZEROES here**
- (iii) we see intervals of increase on $(-2\pi,-3\pi/2)$, $(-\pi/2,\pi/2)$, $(3\pi/2,2\pi)$ ➔ derivative must be **positive here**
- (iv) the opposite is true of intervals of decrease
- (v) intervals of concave up are $(-\pi,0)$ and $(\pi,2\pi)$ ➔ so derivative must be **increasing here**
- (vi) the opposite is true for intervals of concave up



- So the derivative function must look like ➔ **cosine graph**

(A) Derivative of the Sine Function - Graphically

- We will predict the derivative of f(x) = sin(x) from a GRAPHICAL ANALYSIS perspective:
- We will simply sketch 2 cycles
- (i) we see a maximum at $\pi/2$ and $-3\pi/2$ ➔ derivative must have x-intercepts
- (ii) we see intervals of increase on $(-2\pi,-3\pi/2)$, $(-\pi/2,\pi/2)$, $(3\pi/2,2\pi)$ ➔ derivative must be positive on these intervals
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- (iv) intervals of concave up are $(-\pi,0)$ and $(\pi,2\pi)$ ➔ so derivative must increase on these domains
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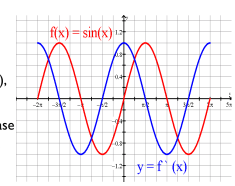

- So the derivative function must look like ➔ the cosine function!!

(A) Derivative of the Sine Function - Technology

- We will predict the what the derivative function of f(x) = sin(x) looks like from our graphing calculator:

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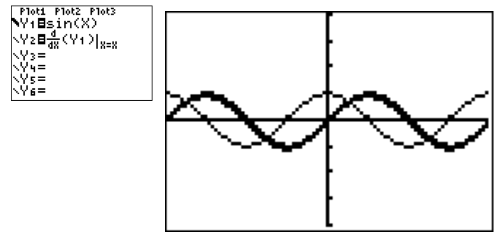
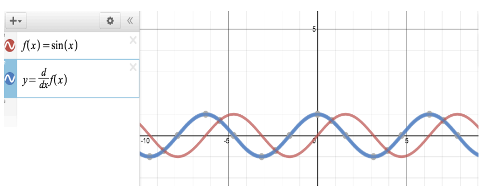


(A) Derivative of the Sine Function - Technology

- We will predict the what the derivative function of f(x) = sin(x) looks like from DESMOs:



(B) Derivative of Sine Function - Algebraically

- We will go back to our limit concepts for an algebraic determination of the derivative of y = sin(x)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)[\cos(h)-1)]+\sin(h)\cos(x)}{h}$$

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x)[\cos(h)-1]}{h}+\lim_{h\to 0}\frac{\sin(h)\cos(x)}{h}$$

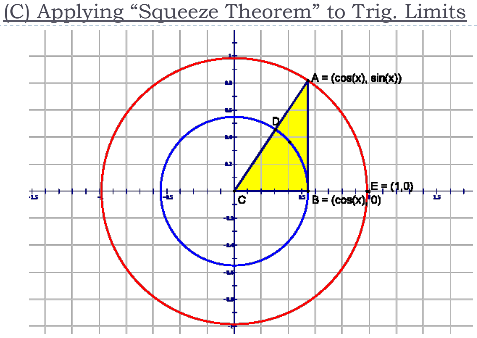
$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}(\sin(x))\times\lim_{h\to 0}\frac{\cos(h)-1}{h}+\lim_{h\to 0}\frac{\sin(h)}{h}\times\lim_{h\to 0}\cos(x)$$

$$\frac{d}{dx}\sin(x) = \sin(x)\times\lim_{h\to 0}\frac{\cos(h)-1}{h}+\cos(x)\times\lim_{h\to 0}\frac{\sin(h)}{h}$$

(B) Derivative of Sine Function - Algebraically

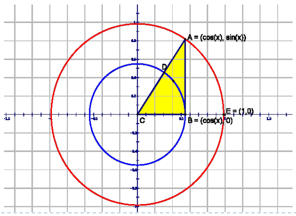
- So we come across 2 special trigonometric limits:
- $\lim_{h\to 0}\frac{\sin(h)}{h}$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h}$
- So what do these limits equal?
- Since we are looking at these ideas from an ALGEBRAIC PERSPECTIVE ➔ We will introduce a new theorem called a Squeeze (or sandwich) theorem ➔ if we that our limit in question lies between two known values, then we can somehow "squeeze" the value of the limit by adjusting/ manipulating our two known values

(C) Applying "Squeeze Theorem" to Trig. Limits



(C) Applying "Squeeze Theorem" to Trig. Limits

- We have sector DCB and sector ACB "squeezing" the triangle ACB
- So the area of the triangle ACB should be "squeezed between" the area of the two sectors



(C) Applying "Squeeze Theorem" to Trig. Limits

- Working with our area relationships (make h = θ)

$$\tfrac{1}{2}(OB)^2(\theta) \le \tfrac{1}{2}(OB)(OA) \le \tfrac{1}{2}(OC)^2(\theta)$$
$$\tfrac{1}{2}\theta \times \cos^2(\theta) \le \tfrac{1}{2}\sin(\theta)\cos(\theta) \le \tfrac{1}{2}\theta \times (1)^2$$
$$\theta\cos^2(\theta) \le \sin(\theta)\cos(\theta) \le \theta$$
$$\frac{\theta\cos^2(\theta)}{\theta\cos(\theta)} \le \frac{\sin(\theta)\cos(\theta)}{\theta\cos(\theta)} \le \frac{\theta}{\theta\cos(\theta)}$$
$$\cos(\theta) \le \frac{\sin(\theta)}{\theta} \le \frac{1}{\cos(\theta)}$$

- We can "squeeze or sandwich" our ratio of sin(h)/h between cos(h) and 1/cos(h)

(C) Applying "Squeeze Theorem" to Trig. Limits

- Now, let's apply the squeeze theorem as we take our limits as h $\to 0^+$ (and since sin(h) has even symmetry, the LHL as h $\to 0^-$)

$$\lim_{h\to 0}\cos(h) \le \lim_{h\to 0}\frac{\sin(h)}{h} \le \lim_{h\to 0}\frac{1}{\cos(h)}$$
$$1 \le \lim_{h\to 0}\frac{\sin(h)}{h} \le 1$$
$$\therefore \lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

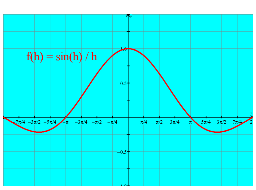
- Follow the link to Visual Calculus - Trig Limits of sin(h)/h to see their development of this fundamental trig limit

(C) Applying "Squeeze Theorem" to Trig. Limits

- Now what about (cos(h) – 1) / h and its limit ➔ we will treat this algebraically

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}$$
$$= \lim_{h\to 0}\frac{(\cos(h)-1)(\cos(h)+1)}{h(\cos(h)+1)}$$
$$= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)}$$
$$= \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)}$$
$$= -1 \times \lim_{h\to 0}\frac{\sin(h)}{h} \times \lim_{h\to 0}\frac{\sin(h)}{(\cos(h)+1)}$$
$$= -1 \times 1 \times \left(\frac{0}{1+1}\right)$$
$$= 0$$

(D) Fundamental Trig. Limits ➔ Graphic and Numeric Verification



x	y
-0.05000	0.99958
-0.04167	0.99971
-0.03333	0.99981
-0.02500	0.99990
-0.01667	0.99995
-0.00833	0.99999
0.00000	undefined
0.00833	0.99999
0.01667	0.99995
0.02500	0.99990
0.03333	0.99981
0.04167	0.99971
0.05000	0.99958

(D) Derivative of Sine Function

- Since we have our two fundamental trig limits, we can now go back and algebraically verify our graphic "estimate" of the derivative of the sine function:

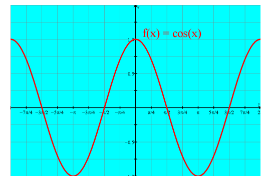
$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$
$$\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0$$
$$\frac{d}{dx}\left(\sin(x)\right) = \sin(x) \times \lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x) \times \lim_{h\to 0}\frac{\sin(h)}{h}$$
$$\frac{d}{dx}\left(\sin(x)\right) = \sin(x) \times 0 + \cos(x) \times 1$$
$$\frac{d}{dx}\left(\sin(x)\right) = \cos(x)$$

(E) Derivative of the Cosine Function

- Knowing the derivative of the sine function, we can develop the formula for the cosine function
- First, consider the graphic approach as we did previously

(E) Derivative of the Cosine Function

- We will predict the what the derivative function of f(x) = cos(x) looks like from our curve sketching ideas:
- We will simply sketch 2 cycles
- (i) we see a maximum at 0, -2π & 2π ➔ derivative must have x-intercepts
- (ii) we see intervals of increase on (-π,0), (π, 2π) ➔ derivative must increase on this intervals
- (iii) the opposite is true of intervals of decrease
- (iv) intervals of concave up are (-3π/2,-π/2) and (π/2 ,3π/2) ➔ so derivative must increase on these domains
- (v) the opposite is true for intervals of concave up
- So the derivative function must look like ➔ some variation of the sine function!!



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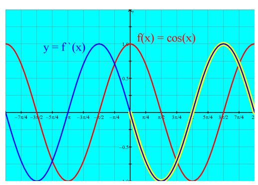
- (iv) intervals of concave up are (-3π/2,-π/2) and (π/2 ,3π/2) ➔ so derivative must increase on these domains
- (v) the opposite is true for intervals of concave up
- So the derivative function must look like ➔ the negative sine function!!



(E) Derivative of the Cosine Function

- Knowing the derivative of the sine function, we can develop the formula for the cosine function
- First, consider the algebraic approach as we did previously
- Recalling our IDENTITIES ➔ cos(x) can be rewritten in TERMS OF SIN(X) as:
- (a) y = sin(pi/2 – x)
- (b) y = sqrt(1 – $\sin^2$(x))

(E) Derivative of the Cosine Function

- Let's set it up algebraically:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sin\left(\frac{\pi}{2}-x\right)\right)$$
$$\frac{d}{dx}(\cos(x)) = \frac{d}{d\left(\frac{\pi}{2}-x\right)}\left(\sin\left(\frac{\pi}{2}-x\right)\right)\times\frac{d}{dx}\left(\frac{\pi}{2}-x\right)$$
$$\frac{d}{dx}(\cos(x)) = \cos\left(\frac{\pi}{2}-x\right)\times(-1)$$
$$\frac{d}{dx}(\cos(x)) = \sin(x)\times -1 = -\sin(x)$$

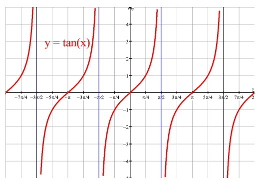
(E) Derivative of the Cosine Function

- Let's set it up algebraically:

$$\frac{d}{dx}(\cos(x)) = \frac{d}{dx}\left(\sqrt{1-\sin^2(x)}\right)$$
$$\frac{d}{dx}(\cos(x)) = \frac{1}{2}\left(1-\sin^2(x)\right)^{-\frac{1}{2}} \bullet -2\sin(x)\cos(x)$$
$$\frac{d}{dx}(\cos(x)) = \frac{1}{2\sqrt{1-\sin^2(x)}} \bullet -2\sin(x)\cos(x)$$
$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\sqrt{1-\sin^2(x)}}$$
$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\sqrt{\cos^2(x)}}$$
$$\frac{d}{dx}(\cos(x)) = \frac{-2\sin(x)\cos(x)}{2\cos(x)}$$
$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

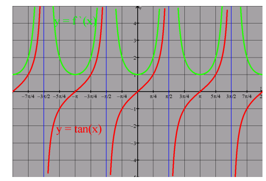
(F) Derivative of the Tangent Function - Graphically

- So we will go through our curve analysis again
- f(x) is constantly increasing within its domain
- f(x) has no max/min points
- f(x) changes concavity from con down to con up at 0,$\pm\pi$
- f(x) has asymptotes at $\pm 3\pi$
- /2, $\pm\pi/2$



(F) Derivative of the Tangent Function - Graphically

- So we will go through our curve analysis again:
- F(x) is constantly increasing within its domain ➔ f `(x) should be positive within its domain
- F(x) has no max/min points ➔ f '(x) should not have roots
- F(x) changes concavity from con down to con up at 0,$\pm\pi$ ➔ f '(x) changes from decrease to increase and will have a min
- F(x) has asymptotes at $\pm 3\pi$ ➔
- /2, $\pm\pi/2$ ➔ derivative should have asymptotes at the same points



(F) Derivative of the Tangent Function - Algebraically

- We will use the fact that tan(x) = sin(x)/cos(x) to find the derivative of tan(x)

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)$$

$$\frac{d}{dx}(\tan(x)) = \frac{\frac{d}{dx}(\sin(x)) \times \cos(x) - \frac{d}{dx}(\cos(x)) \times \sin(x)}{(\cos(x))^2}$$

$$\frac{d}{dx}(\tan(x)) = \frac{\cos(x) \times \cos(x) - (-\sin(x)) \times \sin(x)}{\cos^2 x}$$

$$\frac{d}{dx}(\tan(x)) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$\frac{d}{dx}(\tan(x)) = \frac{1}{\cos^2 x} = \sec^2 x$$

Differentiating with sin(x) & cos(x)

- Differentiate the following
- y = cos(x^2)
- y = $\cos^2$(x)
- y = 3sin(2x)
- y = 6xsin($3x^2$)

- Differentiate the following:

$$y(t) = \sqrt{1 + \cos t + \sin^2 t}$$

$$f(x) = \frac{x^2}{2 - \cos(\pi x)}$$

$$f(y) = y^2 \cos(3y^3)$$

Applications – Tangent Lines

- Find the equation of the tangent line to f(x) = xsin(2x) at the point x = π/4
- What angle does the tangent line to the curve y = f(x) at the origin make with the x-axis if y is given by the equation $y = \frac{1}{\sqrt{3}} \sin 3x$

Applications – Curve Analysis

- Find the maximum and minimum point(s) of the function f(x) = 2cosx + x on the interval (-π,π)
- Find the minimum and maximum point(s) of the function f(x) = xsinx + cosx on the interval (-π/4,π)
- Find the interval in which g(x) = sin(x) + cos(x) is increasing on xER

Applications

- Given

$$g(x)=\begin{cases}\sin x & 0\le x\le \frac{2\pi}{3}\\ ax+b & \frac{2\pi}{3}<x\le 2\pi\end{cases}$$

- (a) for what values of *a* and *b* is g(x) differentiable at 2π/3
- (b) using the values you found for *a* & *b*, sketch the graph of g(x)

(G) Internet Links

- Calculus I (Math 2413) - Derivatives - Derivatives of Trig Functions from Paul Dawkins
- Visual Calculus - Derivative of Trigonometric Functions from UTK
- Differentiation of Trigonometry Functions - Online Questions and Solutions from UC Davis
- The Derivative of the Sine from IEC - Applet

(H) Homework

- Stewart, 1989, Chap 7.2, Q1-5,11
- Handout from Stewart, Calculus: A First Course, 1989, Chap 7.2, Q1&3 as needed, 4-7,9