# Lesson 37 - Second Derivatives, Concavity, Inflection Points

Calculus - Santowski



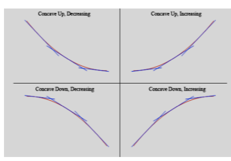
## Lesson Objectives

- 1. Calculate second derivatives of functions
- 2. Define concavity and inflection point
- 3. Test for concavity in a function using the second derivative
- 4. Apply concepts of concavity, second derivatives, inflection points to a real world problem



## (B) New Term – Concave Up

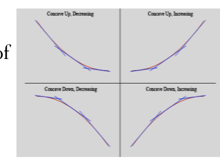
- Concavity is best "defined" with graphs
- (i) "concave up" means in simple terms that the "direction of opening" is upward or the curve is "cupped upward"
- An alternative way to describe it is to visualize where you would draw the tangent lines ➔ you would have to draw the tangent lines "underneath" the curve





## (B) New Term – Concave down

- Concavity is best "defined" with graphs
- (ii) "concave down" means in simple terms that the "direction of opening" is downward or the curve is "cupped downward"
- An alternative way to describe it is to visualize where you would draw the tangent lines ➔ you would have to draw the tangent lines "above" the curve



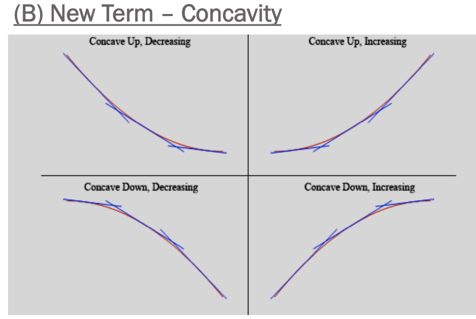


## (B) New Term – Concavity

- In keeping with the idea of concavity and the drawn tangent lines, if a curve is concave up and we were to draw a number of tangent lines and determine their slopes, we would see that the values of the tangent slopes increases (become more positive) as our x-value at which we drew the tangent slopes increase
- This idea of the "increase of the tangent slope is illustrated on the next slides:



## (B) New Term – Concavity



## (C) Calculus & Concavity

- If f ``(x) > 0, then f(x) is concave up
- If f ``(x) < 0, then f(x) is concave down
- If f ``(x) = 0, then f(x) is neither concave nor concave down

- The second derivative also gives information about the "extreme points" or "critical points" or max/mins on the original function (Second Derivative Test):
  - If f `(x) = 0 and f ``(x) > 0, then the critical point is a minimum point (picture $y = x^2$ at x = 0)
  - If f `(x) = 0 and f ``(x) < 0, then the critical point is a maximum point (picture $y = -x^2$ at x = 0)



## (C) Calculus & Concavity

- The first derivative also tells us information about the concavity of a function f(x)

- If f `(x) is increasing on an interval (a,b), then f(x) is concave up on that interval (a,b)
- If f `(x) is decreasing on an interval (a,b), then f(x) is concave down on that interval (a,b)



## (D) Inflection Points & Calculus

- Let f(x) be a differentiable function on an interval (a,b). A point x = a is called an inflection point if the function is continuous at the point and the concavity of the graph changes at that point.

- Using Calculus, the IP can by either f"(a) = 0 or f"(a) does not exist. However, if f"(a) = 0, we should still test on either side of x = a to see IF the concavity changes



## (E) Example (CI)

- Ex 1. Find where the curve $y = 4x^3 - 3x^2 + 1$ is concave up and concave down and determine the co-ordinates of the inflection point(s). Then use this info to sketch the curve

- Ex 2. Find where the curve $y = x^4 - 4x^3 + 5$ is concave up and concave down and determine the co-ordinates of the inflection point(s). Then use this info to sketch the curve



## (E) Example

- Ex 3. Determine the intervals of concavity and inflection points of $f(x) = 3x^5 - 5x^3 + 3$. For this question, you will solve graphically and then verify algebraically



## (G) In Class Examples

- Ex 5. For the following functions, find the intervals of concavity and inflection point(s)

(a) $f(x) = x^2 e^x$

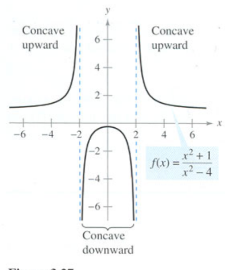
(a) $g(x) = \dfrac{\ln x}{\sqrt{x}}$

(a) $f(x) = \dfrac{x^2+1}{x^2-4}$

(a) $f(x) = \dfrac{6}{x^2+3}$

## Determining Concavity

$$f(x) = \frac{x^2+1}{x^2-4}$$

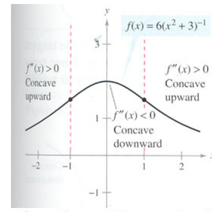



| Interval | $-\infty < x < -2$ | $-2 < x < 2$ | $2 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -3$ | $x = 0$ | $x = 3$ |
| Sign of $f''(x)$ | $f''(-3) > 0$ | $f''(0) < 0$ | $f''(3) > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |



## Determining Concavity

$$f(x) = \frac{6}{x^2+3}$$




Because $f''(x) = 0$ when $x = \pm 1$ and $f''$ is defined on the entire real line, you should test $f''$ in the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. The results are shown in the table and in Figure 3.26.

| Interval | $-\infty < x < -1$ | $-1 < x < 1$ | $1 < x < \infty$ |
|---|---|---|---|
| Test value | $x = -2$ | $x = 0$ | $x = 2$ |
| Sign of $f''(x)$ | $f''(-2) > 0$ | $f''(0) < 0$ | $f''(2) > 0$ |
| Conclusion | Concave upward | Concave downward | Concave upward |



## (F) Inflection Points & BUT......

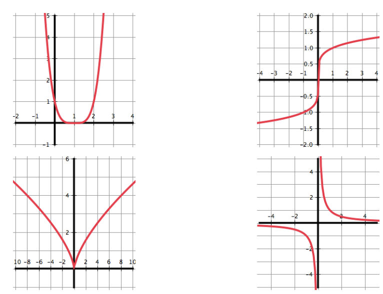
- For the following functions algebraically determine, then graphically verify:

- (i) y"(x)
- (ii) where the inflection points are
- (iii) what their intervals of concavity are

$$f(x) = (x-1)^4$$
$$g(x) = \sqrt[5]{x+2}$$
$$h(x) = (2x-1)^{2/3}$$
$$k(x) = \frac{2}{2-x}$$



## (F) Inflection Points & BUT....





## (F) Inflection Points & BUT......

- Conclusions:
- (a) if f"(a) = 0, then x = a may still NOT be an inflection point
- (b,c) if g'(a) or h'(a) do not exist, concavity may still change at x = a
- (d) concavity may change at an x value which is not in the domain of k(x)

$$f(x) = (x-1)^4$$
$$g(x) = \sqrt[5]{x+2}$$
$$h(x) = (2x-1)^{2/3}$$
$$k(x) = \frac{2}{2-x}$$



## (E) Example (CA – homescreen only)

- Ex 4. For the function $f(x) = \sqrt[3]{x}(x+3)^{\frac{2}{3}}$ find (a) intervals of increase and decrease, (b) local max/min (c) intervals of concavity, (d) inflection point, (e) sketch the graph

- Verify graph using TI-84 graphing features

## Second Derivative Test

- Using the Second Derivative Test to find relative max or mins
- Find the relative extrema of $P(x) = x^3 + x^2 - x + 1$
- Find the relative extrema of $f(x) = -3x^5 + 5x^3$



Using the Second Derivative Test to find relative max or mins

Find the relative extrema of $f(x) = -3x^5 + 5x^3$

Find critical numbers first (what makes first derivative = 0)

$$f'(x) = -15x^4 + 15x^2 = -15x^2(x^2 - 1)$$

So critical numbers are $x = 0, -1, 1$

Using f "(x) = $-60x^3 + 30x$, apply 2nd Derivative test.

| Point on f(x) | (-1, -2) | (1, 2) | (0, 0) |
|---|---|---|---|
| Sign of f"(x) | f " (-1) > 0 | f "(1) < 0 | f "(0) = 0 |
| Conclusion | Concave up so relative min | Concave down so relative max | Test fails |



## Internet Links

- Second Derivatives from P. Dawkins at Lamar U
- Algebra Lab - Second Derivatives & Concavities
- From Monterey Institute for Technology & Education

- On-line Quiz for Fcns & their derivatives
- On-line Quiz #2 for Fcns & their derivatives
- On-line Quiz for Fcns & their derivatives