Lesson 30 - Rates of Change

IBHL Math & Calculus - Santowski

Lesson Objectives

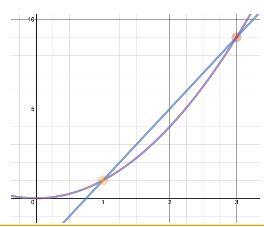
- 1. Calculate an average rate of change
- 2. Estimate instantaneous rates of change using a variety of techniques
- 3. Calculate an instantaneous rate of change using difference quotients and limits
- 4. Calculate instantaneous rates of change numerically, graphically, and algebraically
- 5. Calculate instantaneous rates of change in real world problems

(A) Average Rates of Change

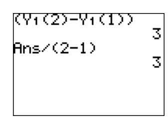
- Given the function $f(x) = x^2$, determine the average rate of change of the function values between x = 1 and x = 2
- (i.e find the slope of the secant line connecting the points (2,4) and (1,1).)
- Sketch to visualize this scenario. Interpret your result!

(A) Average Rates of Change

- Here is the graph of $y = x^2$ and the secant line



- Here is the calculation of the slope from the TI-84



(A) Average Rates of Change

- Then, the interpretation:
- The function values are changing on average 3 units in *y* for every 1 unit in *x* in the interval of *x* = 1 to *x* = 2.
- We can put this into an application ➔ motion

(A) Average Rates of Change: Motion

- If a ball is thrown into the air with a velocity of 30 m/s, its height, *y*, in meters after *t* seconds is given by $y(t)=30t-5t^2$.
- Find the average velocity for the time period between:
- (i) t = 2s and t = 3s
- (ii) t = 2.5s and t = 3s
- (iii) t = 2.9 and t = 3s

(B) Instantaneous Rates of Change

- Now we want to take this concept of rates of change one step further and work with instantaneous rate of change

(B) Instantaneous Rates of Change

- The INSTANTANEOUS rate of change of the function at the given point can be visualized on a graph by the use of a tangent line
- So finding the slope of a tangent line to a curve at a point is the same as finding the rate of change of the curve at that point

(B) Slopes of Tangents ➔ Meaning

- Recall what slopes mean ➔ slope means a rate of change (the change in "rise" given the change in "run")
- Recall the slope formulas:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

$$m = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

(B) Instantaneous Rates of Change

- Given the function $f(x) = x^2$, determine the instantaneous rate of change (in other words the slope of the tangent line to the curve) at the point (1,1). Interpret your result!

(B) Slopes of Tangents – Graphically on TI-84

- Here is how we can visualize (and solve) the problem using the GDC
- We simply ask the calculator to graph the tangent line!!

(B) Estimating Tangent Slopes – Using Secant Slopes

- For the time being we cannot directly use our algebra skills to find the slope of a line if we only know ONE point on the line
- STRATEGY: Let's use secant slopes and make estimates using a secant slope!

(B) Estimating Tangent Slopes – Using Secant Slopes

- The point P(1,3) lies on the curve $y = 4x - x^2$.
- If Q is a second point on the curve, find the slope of the secant PQ if the x-coordinate of Q is:
  - (i) x = 2 (vi) x = 0
  - (ii) x = 1.5 (vii) x = 0.5
  - (iii) x = 1.1 (viii) x = 0.9
  - (iv) x = 1.01 (ix) x = 0.99
  - (v) x = 1.001 (x) x = 0.999

(B) Estimating Tangent Slopes – Using Secant Slopes

- Use the ideas developed so far to determine the instantaneous rate of change of the following functions at the given x-coordinates. Use the TI-84 to help you with carry out algebra:

$$y = x^3 + x - 2 \text{ at } x = 1$$

$$y = e^x + x \text{ at } x = 0$$

$$y = \frac{1}{x+1} \text{ at } x = 2$$

$$y = 2\sin(x)\cos(2x) \quad x = \frac{\pi}{2}$$

Lesson 30 - Rates of Change – Day 2

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Lesson Objectives

- 1. Calculate tangent slopes using limit definitions for instantaneous rates of change
- 2. Calculate tangent slopes & instantaneous rates of change and apply to real world scenarios

Fast Five

- 1. Given the various equations for f(x), evaluate and interpret the following limit:
- (i). $f(x) = x^2 + x$ $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$
- (ii) $f(x) = \sqrt{x+3}$
- (iii) $f(x) = \frac{1}{x-1}$

(A) Review

- We have determined a way to estimate an instantaneous rate of change (or the slope of a tangent line) which is done by means of a series of secant lines such that the secant slope is very close to the tangent slope. We accomplish this "closeness" by simply moving our secant point closer and closer to our tangency point, such that the secant line almost sits on top of the tangent line because the secant point is almost on top of our tangency point.
- Q? Is there an algebraic method that we can use to simplify the tedious approach of calculating secant slopes and get right to the tangent slope??

(B) Internet Links - Applets

- The process outlined in the previous slides is animated for us in the following internet links:
- https://www.geogebratube.org/student/m35986
- https://www.geogebratube.org/student/m145541
- https://www.geogebratube.org/student/m65262

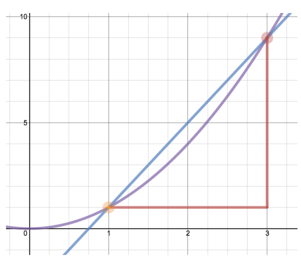
(C) Instantaneous Rates of Change - Algebraic Calculation

- We will adjust the tangent slope formula as follows (to create a more "algebra friendly" formula)

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- See the diagram on the next page for an explanation of the formula

(C) Instantaneous Rates of Change - Algebraic Calculation - Visualization



(E) Example - Determine the Slope of a Tangent Line

- Determine the slope of the tangent line to the curve $f(x) = -x^2 + 3x - 5$ at the point (-4,-33)
- Alternate way to ask the same question:
- Determine the instantaneous rate of change of the function $f(x) = -x^2 + 3x - 5$ at the point (-4,-33)

(D) Instantaneous Rates of Change - Algebraic Calculation

- Find the instantaneous rate of change of: $f(x) = -x^2 + 3x - 5$ at $x = -4$
- We will have to explain the significance of ***h*** in the limit formula (see previous slide)

$$m = \lim_{h \to 0} \frac{f(-4+h) - f(-4)}{h}$$

$$m = \lim_{h \to 0} \frac{\left(-(-4+h)^2 + 3(-4+h) - 5\right) - (-33)}{h}$$

$$m = \lim_{h \to 0} \frac{\left(-(h^2 - 8h + 16) + (-12 + 3h) - 5\right) - (-33)}{h}$$

$$m = \lim_{h \to 0} \frac{(-h^2 + 11h - 33) - (-33)}{h}$$

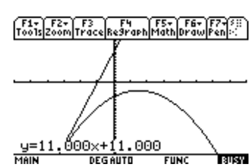
$$m = \lim_{h \to 0} \frac{\cancel{h}(-h + 11)}{\cancel{h}}$$

$$m = \lim_{h \to 0} (-\cancel{h} + 11)$$

$$m = 11$$

(E) Example - Determine the Slope of a Tangent Line

- Now let's confirm this using the TI-84
- So ask the calculator to graph and determine the tangent eqn:



(E) Example - Determine the Slope of a Tangent Line

- Now let's use Wolframalpha to do the limit calculation (recall the function is $f(x) = -x^2 + 3x - 5$) and the limit is

Limit:

$$\lim_{h\to 0} \frac{\left(-(-4+h)^2 + 3(-4+h) - 5\right) - -33}{h} = 11$$

(F) Further Examples - Determine the Slope of a Tangent Line

- Make use of the limit definition of an instantaneous rate of change to determine the instantaneous rate of change of the following functions at the given *x* values:

- $$f(x) = 2x^2 - 5x + 40 \quad at \quad x = 4$$
$$g(x) = 1 + \sqrt{2x} \quad at \quad x = 1$$
$$h(x) = \frac{1}{2-x} \quad at \quad x = 1$$

(G) Applications - Determine the Slope of a Tangent Line

- A football is kicked and its height is modeled by the equation $h(t) = -4.9t^2 + 16t + 1$, where h is height measured in meters and t is time in seconds. Determine the instantaneous rate of change of height at 1 s, 2 s, 3 s.

(G) Applications - Determine the Slope of a Tangent Line

- The solution is:

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t\to 0} \frac{h(1+\Delta t) - h(1)}{\Delta t}$$

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t\to 0} \frac{\left(-4.9(1+\Delta t)^2 + 16(1+\Delta t) + 1\right) - \left(-4.9(1)^2 + 16(1) + 1\right)}{\Delta t}$$

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t\to 0} \frac{\left(-4.9\Delta t^2 + 6.2\Delta t + 12.1\right) - (12.1)}{\Delta t}$$

$$m = \left.\frac{dh}{dt}\right|_{t=1} = \lim_{\Delta t\to 0} \left(-4.9\Delta t + 6.2\right)$$

$$m = \left.\frac{dh}{dt}\right|_{t=1} = 6.2$$

(G) Applications - Determine the Slope of a Tangent Line

- So the slope of the tangent line (or the instantaneous rate of change of height) is 6.2 ➔ so, in context, the rate of change of a distance is called a speed (or velocity), which in this case would be 6.2 m/s at t = 1 sec. ➔ now simply repeat, but use t = 2,3 rather than 1

(H) Applications - Determine the Slope of a Tangent Line

- A business estimates its profit function by the formula $P(x) = x^3 - 2x + 2$ where x is millions of units produced and P(x) is in billions of dollars. Determine the value of the tangent slope at x = ½ and at x = 1½. How would you interpret these values (that correspond to tangent slopes)?

Lesson 30 - Instantaneous Rates of Change at Multiple Points – Day 3

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(A) IRoC as Functions??

- Our change now will be to alter our point of view and let the specific value of *a* or *x* (the point at which we are finding the IRoC) vary (in other words, it will be a variable)
- We will do this as an investigation using two different methods: a graphic/numeric approach and a more algebraic approach

(A) IRoC as Functions??

- We will work with first principles - the tangent concept – and draw tangents to given functions at various points, tabulate results, create scatter-plots and do a regression analysis to determine the equation of the curve of best fit.
- Ex: f(x) = x^2 - 4x - 8 for the interval [-3,8]

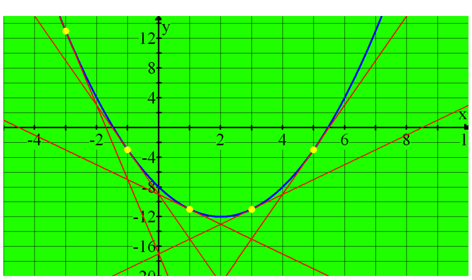
(A) IRoC as Functions??

- Example: $y = x^2 - 4x - 8$. for the interval [-3,8]
- 1. Draw graph.
- 2. Find the tangent slope at x = -3 using the TI-84
- 3. Repeat for x = -2,-1,….,7,8 and tabulate

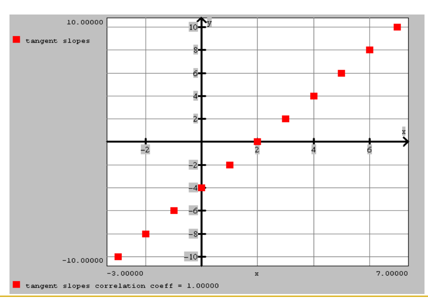
X	-3	-2	-1	0	1	2	3	4	5	6	7	8
Slope	-10	-8	-6	-4	-2	0	2	4	6	8	10	12

- 4. Tabulate data and create scatter-plot
- 5. Find best regression equation

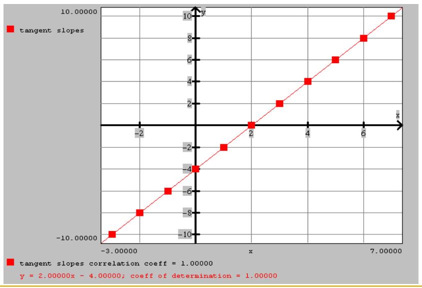
(A) IRoC as Functions??



(A) The Derivative as a Function



(A) The Derivative as a Function



(A) The Derivative as a Function

- Our function equation was f(x) = x^2 - 4x - 8
- Equation generated is y = 2x - 4
- The interpretation of the derived equation is that this "formula" (or equation) will give you the slope of the tangent (or instantaneous rate of change) at every single point x.
- The equation f \`(x) = 2x - 4 is called the derived function, or the derivative of f(x) = x^2 - 4x - 8

(B) The Derivative as a Function - Algebraic

- Given f(x) = x^2 – 4x – 8, we will find the derivative at x = a using our “derivative formula” of

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

- Our one change will be to keep the variable x in the “derivative formula”, since we do not wish to substitute in a specific value like a = 3

(B) The Derivative at a Point - Algebraic

$$f'(3)=\lim_{h\to 0}\frac{f(3+h)-f(3)}{h}$$
$$f'(3)=\lim_{h\to 0}\frac{\left((3+h)^2-4(3+h)-8\right)-\left(3^2-4\cdot 3-8\right)}{h}$$
$$f'(3)=\lim_{h\to 0}\frac{\left(3^2+2\cdot 3\cdot h+h^2-4\cdot 3-4h-8\right)-\left(3^2-4\cdot 3-8\right)}{h}$$
$$f'(3)=\lim_{h\to 0}\frac{2\cdot 3\cdot h+h^2-4h}{h}$$
$$f'(3)=\lim_{h\to 0}\left(2\cdot 3+h-4\right)$$
$$f'(3)=2\cdot 3-4=2$$

(B) The Derivative as a Function - Algebraic

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left((x+h)^2-4(x+h)-8\right)-\left(x^2-4x-8\right)}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{\left(x^2+2xh+h^2-4x-4h-8\right)-\left(x^2-4x-8\right)}{h}$$
$$f'(x)=\lim_{h\to 0}\frac{2xh+h^2-4h}{h}$$
$$f'(x)=\lim_{h\to 0}\left(2x+h-4\right)$$
$$f'(x)=2x-4$$