Lesson 23 – Addition & Subtraction Identities

HL Math - Santowski

Fast Five

- True or False (and justify your responses algebraically, graphically & numerically – but without the use of a calculator)

(a) $\sin\left(x + \dfrac{\pi}{4}\right) = \sin(x) + \sin\left(\dfrac{\pi}{4}\right)$

(b) $\cos\left(\dfrac{\pi}{2} - x\right) = \cos\left(\dfrac{\pi}{2}\right) + \cos(x)$

(A) Six New Identities (GASP!!)

- Here are six new identities that we call the addition & subtraction identities

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$
$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(B) Proving Cosine Subtraction Identity

- We will prove

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

- We will use our unit circle to do so .....
  http://www.cut-the-knot.org/triangle/SinCosFormula.shtml

(C) Proving Sine Addition Identity

- We will prove

$$\sin(A + B) = \sin A \cos B + \sin B \cos A$$

- We will use right triangle trig to do so

- http://www.cut-the-knot.org/triangle/SinCosFormula.shtml

(D) Proving Tan Addition Identity

- You will prove

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

- You will use fundamental trig identities to do so

(E) Using the Addition/Subtraction Identities

- Evaluate each expression

- 1) sin (75°) ➔ using sin (45° + 30°)
- 2) sin (75°) ➔ using sin (120° – 45°)
- 3) cos (345°)

- 4) $\tan\left(\dfrac{11\pi}{12}\right)$

(E) Using the Addition/Subtraction Identities

- Determine the exact value of sin(15°)

- Determine the exact value of cos(-195°)

- Determine the exact value of $\sec\left(\dfrac{5\pi}{12}\right)$

- Determine the exact value of tan(255°)

Pop Quiz

- Find the exact value of

$$\sec\left(\dfrac{13\pi}{12}\right)$$

(E) Using the Addition/Subtraction Identities

- Find each of the following numbers:

- If $\sin A = \dfrac{12}{13}$, for $\dfrac{\pi}{2} < A < \pi$ and $\cos B = -\dfrac{8}{17}$, for $\pi < B < \dfrac{3\pi}{2}$

- 1) Evaluate sin (A + B)
- 2) Evaluate cos (A – B)
- 3) Evaluate tan (A + B)

- 4) If sin(a) = -4/5 for 180°≤a≤270° and if cos(b) = -5/13 for 90°≤b≤180°, evaluate tan(a+b)

(E) Using the Addition/Subtraction Identities

- Simplify the following:

(1) $\cos\left(270° - x\right)$

(2) $\sin\left(x + \dfrac{\pi}{2}\right)$

(3) $\cos\left(x + \pi\right)$

(E) Using the Addition/Subtraction Identities

- We can use the new identities to develop new identities:

- Prove the following: (describe each identity from a transformations perspective as well as a unit circle perspective)

(a) $\cos\left(\pi + x\right) = -\cos x$

(b) $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

(c) $\sin\left(\pi - \theta_{ref}\right) = \sin\left(\theta_{ref}\right)$

(E) Using the Addition/Subtraction Identities

- Find the exact value of:

  (a) $\sin 13° \cos 17° + \sin 17° \cos 13°$

  (b) $\cos 25° \cos 35° - \sin 25° \sin 35°$

(E) Using the Addition/Subtraction Identities

- Prove the following:

  (a) $\cos(x+y)\cos y + \sin(x+y)\sin y = \cos x$

  (b) $1 + \cot x \tan y = \dfrac{\sin(x+y)}{\sin x \cos y}$

Solve for xER

55. $\sin\left(x + \dfrac{\pi}{3}\right) + \sin\left(x - \dfrac{\pi}{3}\right) = 1$

56. $\cos\left(x + \dfrac{\pi}{4}\right) - \cos\left(x - \dfrac{\pi}{4}\right) = 1$

57. $\tan(x + \pi) + 2\sin(x + \pi) = 0$

58. $2\sin\left(x + \dfrac{\pi}{2}\right) + 3\tan(\pi - x) = 0$

Solve for xER

59. $\cos\left(x + \dfrac{\pi}{4}\right) + \cos\left(x - \dfrac{\pi}{4}\right) = 1$

60. $\sin\left(x + \dfrac{\pi}{2}\right) - \cos\left(x + \dfrac{3\pi}{2}\right) = 0$

61. $\tan(x + \pi) - \cos\left(x + \dfrac{\pi}{2}\right) = 0$

62. $\tan(\pi - x) + 2\cos\left(x + \dfrac{3\pi}{2}\right) = 0$

(E) Using the Addition/Subtraction Identities

- We can use the new identities to develop new identities:

- Develop a new identity for:

- (a) sin(2x)
- (b) cos(2x)
- (c) tan(2x)

Linear Combinations of Sine and Cosine

A **sum of multiples** of two functions is called a **linear combination** of the two functions.

Thus, if $f(x) = a \sin x + b \cos x$, then $f(x)$ is a linear combination of $\sin x$ and $\cos x$.

We are going to see an example of the general principle that every linear combination of $\sin x$ and $\cos x$ is a sinusoidal function.

Let $f(x) = 3 \sin x + 4 \cos x$. Show that f is a sinusoidal function. Find the amplitude, period and phase shift.

The first step is to find the square root of the sum of the squares of the coefficients. In this case, that is $\sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. Factor 5 out of the expression to get $f(x) = 5\left(\frac{3}{5}\sin x + \frac{4}{5}\cos x\right)$

Next, regard the coefficient of $\sin x$ as the cosine of some angle ϕ and the coefficient of $\cos x$ as the sine of the same angle ϕ. We can make this assumption since the sum of the squares of the coefficients equals 1 as does the sum of the squares of $\cos \phi$ and $\sin \phi$ for any angle ϕ.

This gives $f(x) = 5(\cos \phi \sin x + \sin \phi \cos x)$

Using the sum formula for sine, this can be re-written $f(x) = 5\sin(x + \phi)$.

The amplitude is 5, the period is 2π and the phase shift is $-\phi$.

What is the value of ϕ ?

Since both the sine and the cosine of ϕ are positive, ϕ is an angle in quadrant I. So we can calculate $\phi = \arccos\left(\frac{3}{5}\right)$ or alternately, $\phi = \arcsin\left(\frac{4}{5}\right)$. So $\phi = 0.9273$ radians or $53.1°$.