Lesson 17 – Introducing and Applying Base *e*.

IBHL1 Math - Santowski

FAST FIVE

- Go to the following DESMOS interactive graph from the lesson notes page and play with it to the end that you can explain what is going on.
- CONTEXT → \$1000 invested at 6% p.a compounded *n* times per year.

(A) FAST FIVE

- 1. Given the geometric series defined as $S_n = \sum_{k=0}^{n} \left(\frac{1}{k!}\right)$
- (a) List the first 5 terms of the series
- (b) Find the first five partial sums
- (c) evaluate the series if n = 20
- (d) evaluate the series as n → ∞
- 2. Evaluate $f(x) = \left(1 + \frac{1}{x}\right)^x$ for $x = \{1,2,5,15,50\}$

Lesson Objectives

- (1) Investigate a new base to use in exponential applications
- (2) Understand WHEN the use of base e is appropriate
- (3) Apply base e in word problems and equations

Lesson Objective #1

- (1) Investigate a new base to use in exponential applications
- GIVEN: the formula for working with compound interest → $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- Determine the value after 2 years of a \$1000 investment under the following compounding conditions:

(A) Working with Compounding Interest

- Determine the value after 2 years of a \$1000 investment under the following investing conditions:
- (a) Simple interest of 10% p.a
- (b) Compound interest of 10% pa compounded annually
- (c) Compound interest of 10% pa compounded semi-annually
- (d) Compound interest of 10% pa compounded quarterly
- (e) Compound interest of 10% pa compounded daily
- (f) Compound interest of 10% pa compounded n times per year

(B) Introducing Base *e*

- Take $1000 and let it grow at a rate of 10% p.a. Then determine value of the $1000 after 2 years under the following compounding conditions:
- (i) compounded annually =1000(1 + .1/1)$^{(2x1)}$ = 1210
- (ii) compounded quarterly =1000(1 + 0.1/4)$^{(2x4)}$ = 1218.40
- (iii) compounded daily =1000(1 + 0.1/365)$^{(2x365)}$ = 1221.37
- (iv) compounded *n* times per year =1000(1+0.1/n)$^{(2xn)}$ = ????

(B) Introducing Base *e*

- So we have the expression 1000(1 + 0.1/n)$^{(2n)}$
- Now what happens as we increase the number of times we compound per annum ⇒ i.e. n →∞ ?? (that is … come to the point of **compounding continuously**)
- So we get the idea of an "end behaviour again:

$$1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)} \quad \text{as } n\to\infty$$

(B) Introducing Base *e*

- Now let's rearrange our function
- use a simple substitution ➔ let 0.1/n = 1/x
- Therefore, 0.1x = n ➔ so then $1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}$ as $n\to\infty$

becomes $1000\times\left(1+\frac{1}{x}\right)^{(x\times 0.1\times 2)}$ as $x\to\infty$

- Which simplifies to $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$

(B) Introducing Base *e*

- So we see a special end behaviour occurring:

$$1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2} \quad \text{as } x\to\infty$$

- Now we will isolate the "base" of $\left(1+\frac{1}{x}\right)^{x}$
- We can evaluate the limit a number of ways ➔ graphing or a table of values.

(B) Introducing Base *e*

- So we see a special "end behaviour" occurring:

$$1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2} \quad \text{as } x\to\infty$$

- We can evaluate the "base" $\left(1+\frac{1}{x}\right)^{x}$ by graphing or a table of values.
- In either case, $e=\left(1+\frac{1}{x}\right)^{x}$ as $x\to\infty$
- where *e* is the natural base of the exponential function

(B) Introducing Base *e*

- So our original formula $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$ as $x\to\infty$ now becomes A = 1000**e**$^{0.1x2}$ where the 0.1 was the interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment (so A = Pe^{rt}) ➔ so our value becomes $1221.40
- And our general equation can be written as A = Pe^{rt} where P is the original amount, r is the annual growth rate and t is the length of time in years

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate
- Recall our original question ➔ Determine the value after 2 years of a $1000 investment under the following investing conditions:
- (a) Compound interest of 10% pa compounded annually
- (b) Compound interest of 10% pa compounded semi-annually
- (c) Compound interest of 10% pa compounded quarterly
- (d) Compound interest of 10% pa compounded daily
- All these examples illustrate **DISCRETE** changes rather than **CONTINUOUS** changes.

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate
- So our original formula $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$ as $x\to\infty$ now becomes A = $1000e^{0.1x2}$ where the 0.1 was the annual interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment ➔ BUT RECALL WHY we use "n" and what it represents …….
- Note that in this example, the growth happens ***continuously*** (i.e the idea that n→∞)

Lesson Objective #3

- (3) Apply base ***e*** in word problems and equations

(C) Working With Exponential Equations in Base **e**

- (i) Solve the following equations:

$$(i)\ e^{x^2-x}=e^2 \qquad (ii)\ \left(e^x\right)^2=\sqrt{e^{x+2}}$$

$$(ii)\ e^{-x^2}=\left(\frac{1}{e}\right)^x \qquad (iv)\ e^{2x\text{-}1}=\frac{1}{e^{3x+1}}$$

$$(v)\ e^x=4 \qquad (vi)\ e^x=-5$$

$$(vii)\ e^x=1-x \qquad (viii)\ e^{2x}+6=5e^x$$

(C) Working with A = Pe^{rt}

- So our formula for situations featuring continuous change becomes A = Pe^{rt} ➔ P represents an initial amount, r the annual growth/decay rate and t the number of years
- In the formula, if r > 0, we have exponential growth and if r < 0, we have exponential decay

(C) Examples

- (i) I invest $10,000 in a funding yielding 12% p.a. compounded continuously.
  - (a) Find the value of the investment after 5 years.
  - (b) How long does it take for the investment to triple in value?
- (ii) The population of the USA can be modeled by the eqn P(t) = $227e^{0.0093t}$, where P is population in millions and t is time in years since 1980
  - (a) What is the annual growth rate?
  - (b) What is the predicted population in 2015?
  - (c) What assumptions are being made in question (b)?
  - (d) When will the population reach 500 million?

(C) Examples

- A population starts with 500 viruses that grows to a population of 600 viruses in 2 days.

- (a) Assuming LINEAR GROWTH, write a linear model ($u_n = u_1 + (n-1)d$) for population growth
- (b) Assuming DISCRETE EXPONENTIAL GROWTH, write an exponential model ($u_n = u_1 r^{n-1}$) for population growth
- (c) Assuming CONTINUOUS EXPONENTIAL GROWTH, write an exponential model ($A = A_0 e^{rt}$) for population growth.

- Use your models to predict the number of viruses in one month.
- Explain WHY it is important to work with APPROPRIATE and ACCURATE models, given the recent Ebola outbreak in West Africa

(C) Examples

- (iii) A certain bacteria grows according to the formula $A(t) = 5000e^{0.4055t}$, where t is time in hours.
  - (a) What will the population be in 8 hours
  - (b) When will the population reach 1,000,000

- (iv) The function $P(t) = 1 - e^{-0.0479t}$ gives the percentage of the population that has seen a new TV show t weeks after it goes on the air.
  - (a) What percentage of people have seen the show after 24 weeks?
  - (b) Approximately, when will 90% of the people have seen the show?
  - (c) What happens to P(t) as t gets infinitely large? Why? Is this reasonable?

(F) Examples with Applications

- Two populations of bacteria are growing at different rates. Their populations at time t are given by $P_1(t) = 5^{t+2}$ and $P_2(t) = e^{2t}$ respectively. At what time are the populations the same?