Lesson 16- Solving Exponential Equations

IB Math HL - Santowski

Lesson Objectives

- (1) Establish a context for the solutions to exponential equations
- (2) Review & apply strategies for solving exponential eqns:
  - (a) guess and check
  - (b) graphic
  - (c) algebraic
    - (i) rearrange eqn into equivalent bases
    - (ii) isolate parent function and apply inverse

Lesson Objective #1 – Context for Exponential Equations

- (1) Establish a context for the solutions to exponential equations
- (a) population growth
- (b) decay

(A) Context for Equations

- Write and then solve equations that model the following scenarios:
- ex. 1 The model $P(t) = P_0 2^{t/d}$ can be used to model bacterial growth. Given that a bacterial strain doubles every 30 minutes, how much time is required for the bacteria to grow from an initial 100 to 25,600?
- ex 2. The number of bacteria in a culture doubles every 2 hours. The population after 5 hours is 32,000. How many bacteria were there initially?

Lesson Objective #2 – Solving Eqns

- Review & apply strategies for solving exponential eqns:
  - (a) guess and check
  - (b) graphic
  - (c) algebraic
    - (i) rearrange eqn into equivalent bases
    - (ii) isolate parent function and apply inverse

(A) Solving Strategy #1 – Guess and Check

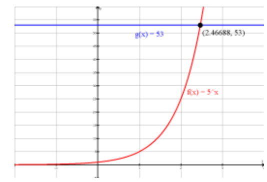
- Solve $5^x = 53$ using a guess and check strategy
- Solve $2^x = 3$ using a guess and check strategy

(A) Solving Strategy #1 – Guess and Check

- Solve $5^x = 53$ using a guess and check strategy
- we can simply "guess & check" to find the right exponent on 5 that gives us 53 ➔ we know that $5^2 = 25$ and $5^3 = 125$, so the solution should be somewhere closer to 2 than 3
- Solve $2^x = 3$ using a guess and check strategy
- we can simply "guess & check" to find the right exponent on 2 that gives us 3 ➔ we know that $2^1 = 2$ and $2^2 = 4$, so the solution should be somewhere between to 1 and 2

(B) Solving Strategy #2 – Graphic Solutions

- Going back the example of $5^x = 53$, we always have the graphing option
- We simply graph $y_1 = 5^x$ and simultaneously graph $y_2 = 53$ and look for an intersection point (2.46688, 53)



(B) Solving Strategy #2 – Graphic Solutions

- Solve the following equations graphically.
- (i) $8^x = 2^{x+1}$
- (ii) $3^x = 53$
- (iii) $2^x = 3$
- (iv) $2^{4x-1} = 3^{1-x}$

(C) Solving Strategy #3 – Algebraic Solutions

- We will work with 2 algebraic strategies for solving exponential equations:
  - (a) Rearrange the equations using various valid algebraic manipulations to either (i) make the bases equivalent or (ii) make the exponents equivalent
  - (b) Isolate the parent exponential function and apply the inverse function to "unexponentiate" the parent function

(D) Algebraic Solving Strategies – Math Principles

- MATH PRINCIPLE ➔ PROPERTIES OF EQUALITIES
- If two powers are equal and they have the same base, then the exponents must be the same
- ex. if $b^x = a^y$ and $a = b$, then $x = y$.
- If two powers are equal and they have the same exponents, then the bases must be the same
- ex. if $b^x = a^y$ and $x = y$, then $a = b$.

(E) Solving Strategies – Algebraic Solution #1

- This prior observation sets up our general equation solving strategy => get both sides of an equation expressed in the same base
- ex. Solve and verify the following:
- (a) $(½)^x = 4^{2-x}$ (b) $3^{y+2} = 1/27$
- (c) $(1/16)^{2a-3} = (1/4)^{a+3}$ (d) $3^{2x} = 81$
- (e) $5^{2x-1} = 1/125$ (f) $36^{2x+4} = \sqrt{(1296^x)}$

(E) Solving Strategies – Algebraic Solution #1

- The next couple of examples relate to composed functions ➔quadratic fcns composed with exponential fcns:
- Ex: Let $f(x) = 2^x$ and let $g(x) = x^2 + 2x$, so solve $fog(x) = ½$
- Ex: Let $f(x) = x^2 – x$ and let $g(x) = 2^x$, so solve $fog(x) = 12$

(E) Solving Strategies – Algebraic Solution #1

- The next couple of examples relate to composed functions ➔quadratic fcns composed with exponential fcns:
- Ex: Let $f(x) = 2^x$ and let $g(x) = x^2 + 2x$, so solve $fog(x) = ½$ ➔ i.e. Solve $2^{x^2+2x} = ½$
- Ex: Let $f(x) = x^2 – x$ and let $g(x) = 2^x$, so solve $fog(x) = 12$ ➔ i.e. Solve $2^{2x} - 2^x = 12$

(E) Solving Strategies – Assorted Examples

- Solve the following for x using the most appropriate method:
- (a) $2^x = 8$ (b) $2^x = 1.6$
- (c) $2^x = 11$ (d) $2^x = 32^{2x-2}$
- (e) $2^{4x+1} = 8^{1-x}$ (f) $2^{x^2-4} = 8^x$
- (g) $2^{3x+2} = 9$ (h) $3(2^{2x-1}) = 4^{-x}$
- (i) $2^{4y+1} – 3^y = 0$

Lesson Objective #1 – Context for Exponential Equations

- (1) Establish a context for the solutions to exponential equations

(F) Examples with Applications

- Example 1 ➔ Radioactive materials decay according to the formula $N(t) = N_0(1/2)^{t/h}$ where N_0 is the initial amount, t is the time, and h is the half-life of the chemical, and the (1/2) represents the decay factor. If Radon has a half life of 25 days, how long does it take a 200 mg sample to decay to 12.5 mg?

(F) Examples with Applications

- Example 2 ➔ A financial investment grows at a rate of 6%/a. How much time is required for the investment to double in value?
- Example 3 ➔ A financial investment grows at a rate of 6%/a but is compounded monthly. How much time is required for the investment to double in value?

(F) Examples with Applications

- Two populations of bacteria are growing at different rates. Their populations at time t are given by $P_1(t) = 5^{t+2}$ and $P_2(t) = e^{2t}$ respectively. At what time are the populations the same?

(F) Examples with Applications

- ex 1. Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours, determine how much caffeine is in Mr. S.'s body at class-time (1:10pm). Then determine how much time passes before I have 30 mg of caffeine in my body.
- ex 2. The value of the Canadian dollar , at a time of inflation, decreases by 10% each year. What is the half-life of the Canadian dollar?

(F) Examples with Applications

- ex 3. The half-life of radium-226 is 1620 a. Starting with a sample of 120 mg, after how many years is only 40 mg left?
- ex 4. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.

(F) Examples with Applications

- ex 5. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost
- (a) An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim.
- (b) If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up?
- (c) A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle?

(F) Examples with Applications

- Ex 1. An investment of $1,000 grows at a rate of 5% p.a. compounded annually. Determine the first 5 terms of a geometric sequence that represents the value of the investment at the end of each compounding period.
- Ex 2. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.

(F) Examples with Applications

- Write and then solve equations that model the following scenarios:
- Ex 1. 320 mg of iodine-131 is stored in a lab for 40d. At the end of this period, only 10 mg remains.
  - (a) What is the half-life of I-131?
  - (b) How much I-131 remains after 145 d?
  - (c) When will the I-131 remaining be 0.125 mg?
- Ex 2. Health officials found traces of Radium F beneath P044. After 69 d, they noticed that a certain amount of the substance had decayed to $1/\sqrt{2}$ of its original mass. Determine the half-life of Radium F