Lesson 13 – Quadratic & Polynomial Equations & Complex Numbers

HL1 Math - Santowski

Lesson Objectives

- Find and classify all real and complex roots of a quadratic equation
- Understand the "need for" an additional number system
- Add, subtract, multiply, divide, and graph complex numbers
- Find and graph the conjugate of a complex number

Fast Five

- STORY TIME.....
- http://mathforum.org/johnandbetty/frame.htm

(A) Introduction to Complex Numbers

- Solve the equation $x^2 - 1 = 0$
- We can solve this many ways (factoring, quadratic formula, completing the square & graphically)
- In all methods, we come up with the solution $x = \pm 1$, meaning that the graph of the quadratic (the parabola) has 2 roots at $x = \pm 1$.
- Now solve the equation $x^2 + 1 = 0$

(A) Introduction to Complex Numbers

- Now solve the equation $x^2 + 1 = 0$
- The equation $x^2 = -1$ has no roots because you cannot take the square root of a negative number.
- Long ago mathematicians decided that this was too restrictive.
- They did not like the idea of an equation having no solutions -- so they invented them.
- They proved to be very useful, even in practical subjects like engineering.

(A) Introduction to Complex Numbers

- Consider the general quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$.
- The usual formula obtained by ``completing the square'' gives the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are "happy".

(A) Introduction to Complex Numbers

- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are happy.
- If $b^2 < 4ac$ (or if $b^2 - 4ac < 0$) then the number under the square root is negative and you would say that the equation has no solutions.
- In this case we write $b^2 - 4ac = (-1)(4ac - b^2)$ and $4ac - b^2 > 0$. So, in an obvious formal sense, $x = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a}$
- and now the only `meaningless' part of the formula is $\sqrt{-1}$

(A) Introduction to Complex Numbers

- So we might say that any quadratic equation either has ``real'' roots in the usual sense or else has roots of the form $p \pm q\sqrt{-1}$ where p and q belong to the real number system .
- The expressions $p \pm q\sqrt{-1}$ do not make any sense as real numbers, but there is nothing to stop us from playing around with them as **symbols** as $p + qi$ (but we will use $a + bi$)
- We call these numbers complex numbers; the special number i is called an imaginary number, even though i is just as ``real'' as the real numbers and complex numbers are probably simpler in many ways than real numbers.

(B) Using Complex Numbers → Solving Equations

- Note the difference (in terms of the expected solutions) between the following 2 questions:
- Solve $x^2 + 2x + 5 = 0$ where $x \in R$
- Solve $x^2 + 2x + 5 = 0$ where $x \in C$

(B) Using Complex Numbers → Solving Equations

- Solve the following quadratic equations where $x \in C$
- Simplify all solutions as much as possible
- Rewrite the quadratic in factored form
- $x^2 - 2x = -10$
- $3x^2 + 3 = 2x$
- $5x = 3x^2 + 8$
- $x^2 - 4x + 29 = 0$
- What would the "solutions" of these equations look like if $x \in R$

(B) Using Complex Numbers → Solving Equations

- Solve the following quadratic equations where $x \in C$
- $x^2 - 2x = -10$
- $3x^2 + 3 = 2x$
- $5x = 3x^2 + 8$
- $x^2 - 4x + 29 = 0$
- Now verify your solutions algebraically!!!

(B) Using Complex Numbers → Solving Equations

- One root of a quadratic equation is $2 + 3i$
- (a) What is the other root?
- (b) What are the factors of the quadratic?
- (c) If the y-intercept of the quadratic is 6, determine the equation in factored form and in standard form.

(B) Solving Polynomials if x ε **C**

- Let's expand polynomials to cubics & quartics:
- Factor and solve $3 - 2x^2 - x^4 = 0$ if x ε **C**
- Factor and solve $3x^3 - 7x^2 + 8x - 2 = 0$ if x ε **C**
- Factor and solve $2x^3 + 14x - 20 = 9x^2 - 5$ if x ε **C**
- Now write each polynomial as a product of its factors
- Explain the graphic significance of your solutions for x

(B) Solving if x ε **C** – Solution to Ex 1

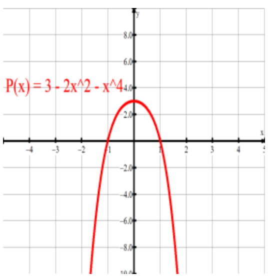
- Factor and solve $3 - 2x^2 - x^4 = 0$ if x ε **C** and then write each polynomial as a product of its factors
- Solutions are x = ± 1 and x = $\pm i\sqrt{3}$
- So rewriting the polynomial in factored form (over the reals) is $P(x) = -(x^2 + 3)(x - 1)(x + 1)$ and over the complex numbers: $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

(B) Solving if x ε **C** – Graphic Connection

- With $P(x) = 3 - 2x^2 - x^4$, we can now consider a graphic connection, given that
  $P(x) = -(x^2 + 3)(x - 1)(x + 1)$

  or given that

  $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

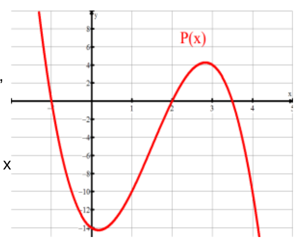


(C) Fundamental Theorem of Algebra

- The fundamental theorem of algebra can be stated in many ways:
  - (a) If P(x) is a polynomial of degree *n* then P(x) will have exactly *n* zeroes (real or complex), some of which may repeat.
  - (b) Every polynomial function of degree $n \geq 1$ has exactly *n* complex zeroes, counting multiplicities
  - (c) If P(x) has a nonreal root, *a*+*b***i**, where $b \neq 0$, then its conjugate, *a*–*b***i** is also a root
  - (d) Every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors
- What does it all mean ➔ we can solve EVERY polynomial (it may be REALLY difficult, but it can be done!)

(D) Using the FTA

- Write an equation of a polynomial whose roots are x = 1, x = 2 and x = ¾
- Write the equation of the polynomial whose roots are 1, -2, -4, & 6 and a point (-1, -84)
- Write the equation of a polynomial whose roots are x = 2 (with a multiplicity of 2) as well as x = -1 $\pm \sqrt{2}$
- Write the equation of a polynomial whose graph is given:



(D) Using the FTA

- Given that 1 – 3i is a root of $x^4 - 4x^3 + 13x^2 - 18x - 10 = 0$, find the remaining roots.
- Write an equation of a third degree polynomial whose given roots are 1 and *i*. Additionally, the polynomial passes through (0,5)
- Write the equation of a quartic wherein you know that one root is 2 – *i* and that the root x = 3 has a multiplicity of 2.

(E) Further Examples

- The equation $x^3 - 3x^2 - 10x + 24 = 0$ has roots of 2, h, and k. Determine a quadratic equation whose roots are $h - k$ and hk.
- The 5th degree polynomial, f(x), is divisible by x^3 and $f(x) - 1$ is divisible by $(x - 1)^3$. Find f(x).
- Find the polynomial p(x) with integer coefficients such that one solution of the equation p(x)=0 is $1+\sqrt{2}+\sqrt{3}$.

(E) Further Examples

- Start with the linear polynomial: $y = -3x + 9$. The x-coefficient, the root and the intercept are -3, 3 and 9 respectively, and these are in arithmetic progression. Are there any other linear polynomials that enjoy this property?
- What about quadratic polynomials? That is, if the polynomial $y = ax^2 + bx + c$ has roots r_1 and r_2 can a, r_1, b, r_2 and c be in arithmetic progression?