Lesson 12 – Working with Polynomial Equations

HL1 Math - Santowski

Lesson Objectives

- Mastery of the factoring of polynomials using the algebraic processes of synthetic division
- Mastery of the algebraic processes of solving polynomial equations by factoring (Factor Theorem & Rational Root Theorem)
- Mastery of the algebraic processes of solving polynomial inequalities by factoring (Factor Theorem & Rational Root Theorem)

(A) FAST FIVE

- Factor $x^2 - x - 2$
- Explain what is meant by the term "factor of a polynomial"
- Explain what is meant by the term "root of a polynomial"
- Divide $x^3 - x^2 - 14x + 24$ by $x - 2$
- Divide $x^3 - x^2 - 14x + 24$ by $x + 3$

(A) Roots & Factors

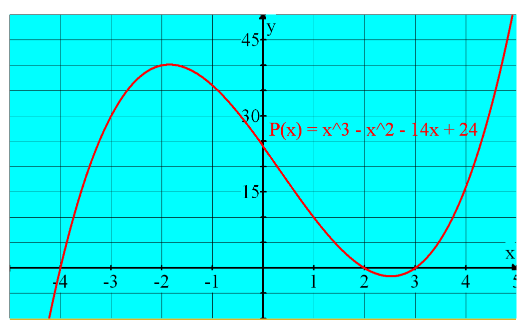
- In our work with quadratics, we saw the "factored" form or "intercept" form of a quadratic equation/expression
- i.e. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$ ➔ factored form of eqn
- So when we solve $f(x) = 0$ ➔ $0 = (x - 2)(x + 1)$, we saw that the zeroes/x-intercepts/roots were $x = 2$ and $x = -1$
- So we established the following connection:
- Factors ➔ $(x - 2)$ and $(x - 1)$
- Roots ➔ $x = 2$ and $x = 1$
- So we will now reiterate the following connections:
- If $(x - R)$ is a ***factor*** of $P(x)$, then $x = R$ is ***root*** of $P(x)$

  AND THE CONVERSE
- If $x = R$ is a ***root*** of $P(x)$, then $(x - R)$ is a ***factor*** of $P(x)$

(B) Review

- Divide $x^3 - x^2 - 14x + 24$ by $x - 2$ and notice the remainder
- Then evaluate $P(2)$. What must be true about $(x - 2)$?
- Divide $x^3 - x^2 - 14x + 24$ by $x + 3$ and notice the remainder
- Then evaluate $P(-3)$. What must be true about $(x+3)$?
- Now graph $f(x) = x^3 - x^2 - 14x + 24$ and see what happens at $x = 2$ and $x = -3$
- So our conclusion is: $x - 2$ is a factor of $x^3 - x^2 - 14x + 24$, whereas $x + 3$ is not a factor of $x^3 - x^2 - 14x + 24$

(B) Review – Graph of $P(x) = x^3 - x^2 - 14x + 24$



(C) The Factor Theorem

- State the Factor Theorem

(C) The Factor Theorem

- We can use the ideas developed in the review to help us to draw a connection between the polynomial, its factors, and its roots.
- What we have seen in our review are the key ideas of the Factor Theorem - in that if we know a root of an equation, we know a factor and the converse, that if we know a factor, we know a root.
- The Factor Theorem is stated as follows: ***x - a* is a factor of *f(x)* if and only if *f(a) = 0*.** To expand upon this idea, we can add that ***ax - b* is a factor of *f(x)* if and only if *f(b/a) = 0*.**
- Working with polynomials, (x + 1) is a factor of $x^2 + 2x + 1$ because when you divide $x^2 + 2x + 1$ by x + 1 you get a 0 remainder and when you substitute x = -1 into $x^2 + 2x + 1$, you get 0

(D) Examples

- ex 1. Show that x - 2 is a factor of $x^3 - 7x + 6$
- ex. 2. Show that -2 is a root of $2x^3 + x^2 - 2x + 8 = 0$. Find the other roots of the equation. (Show with GDC)
- ex. 3. Factor $x^3 + 1$ completely
- ex. 4. Factor $x^3 – 1$ completely
- ex. 4. Is x - √2 a factor of $x^4 – 5x^2 + 6$?

(D) Further Examples

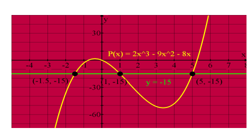
- Ex 1 ➔ Factor $P(x) = 2x^3 – 9x^2 + 7x + 6$ & then solve P(x) = 0
- Hence, sketch y = |P(x)|
- Solve $2x^3 – 9x^2 + 7x + 6 < 0$
- Ex 2 ➔ Factor $3x^3 – 7x^2 + 8x – 2$ & then solve P(x) = 0
- Hence, sketch y = |P(x)|
- Solve $3x^3 – 7x^2 + 8x \geq 2$
- Ex 3 ➔ Factor & solve $f(x) = 3x^3 + x^2 – 22x – 24$
- Hence, sketch y = |P(x)|
- Solve $-3x^3 \leq x^2 – 22x – 24$

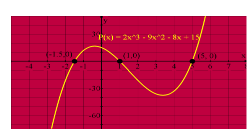
(D) Further Examples: Systems

- ex.1 Solve $2x^3 – 9x^2 – 8x = -15$ and then show on a GDC
- ie. Solve the system $\begin{cases} y = 2x^3 - 9x^2 - 8x \\ y = -15 \end{cases}$
- ex 2. Solve $2x^3 + 14x – 20 = 9x^2 – 5$ and then show on a GDC
- ie. Solve the system $\begin{cases} y = 2x^3 + 14x - 20 \\ y = 9x^2 - 5 \end{cases}$

(D) Further Examples - Solutions

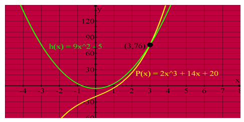
- Solve $2x^3 – 9x^2 – 8x = -15$ and then show on a GDC
- Now graph both
- $g(x) = 2x^3 – 9x^2 – 8x$ and then h(x) = -15 and find intersection
- OR graph:
- $f(x) = 2x^3 – 9x^2 – 8x + 15$





(D) Further Examples - Solutions

- Solve $2x^3 + 14x - 20 = 9x^2 - 5$ and then show on a GDC
- Explain that different solution sets are possible depending on the number set being used (real or complex)

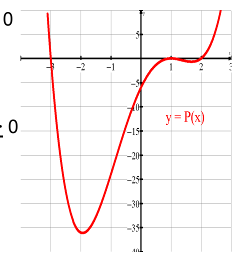


(D) Further Examples - Quartics

- Solve $x^4 - x^3 - 7x^2 + 13x - 6 = 0$
- Hence solve $x^4 - x^3 + 13x \geq 6 + 7x^2$
- Hence, sketch a graph of $P(x) = | x^4 - x^3 - 7x^2 + 13x - 6 |$
- Hence, sketch a graph of the reciprocal of $x^4 - x^3 - 7x^2 + 13x - 6$

(D) Further Examples - Solutions

- Solve $x^4 - x^3 - 7x^2 + 13x - 6 = 0$
- $P(x) = 0 = (x - 1)^2(x + 3)(x - 2)$
- Solve $x^4 - x^3 - 7x^2 + 13x - 6 \geq 0$
- So $P(x) \geq 0$ on $[-\infty,-3)$ or $(2, \infty)$

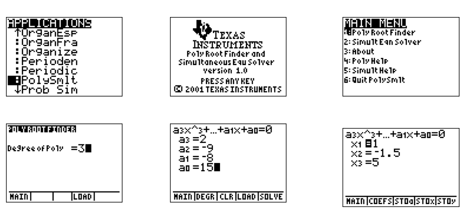


(E) Solving & Factoring on the TI-84

- Factor & solve the following on your calculator:
- $2x^3 - 9x^2 - 8x = -15$
- $0 = 2x^3 - 9x^2 + 7x + 6$
- $x^4 - x^3 - 7x^2 + 13x - 6 = 0$

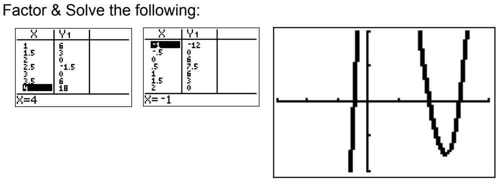
(E) Solving & Factoring on the TI-84

- Solve $2x^3 - 9x^2 - 8x = -15$ turn it into a "root" question ➔ i.e Solve $P(x) = 0$ ➔ Solve $0 = 2x^3 - 9x^2 - 8x + 15$



(E) Solving & Factoring on the TI-84

- Factor & Solve the following:



- $0 = 2x^3 - 9x^2 + 7x + 6$ ➔ roots at x = -0.5,2,3 ➔ would imply factors of $(x - 2)$, $(x - 3)$ and $(x + ½)$ ➔ $P(x) = 2(x + ½)(x - 2)(x - 3)$
- So when factored $P(x) = (2x + 1)(x - 2)(x - 3)$

(F) Multiplicity of Roots

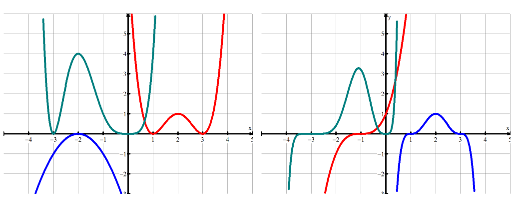
- Factor the following polynomials:
- $P(x) = x^2 – 2x – 15$
- $P(x) = x^2 – 14x + 49$
- $P(x) = x^3 + 3x^2 + 3x + 1$
- Now solve each polynomial equation, P(x) = 0
- Solve $0 = 5(x + 1)^2(x – 2)^3$
- Solve $0 = x^4(x – 3)^2(x + 5)$
- Solve $0 = (x + 1)^3(x – 1)^2(x – 5)(x + 4)$

(F) Multiplicity of Roots

- If *r* is a zero of a polynomial and the exponent on the factor that produced the root is *k*, $(x – r)^k$, then we say that *r* has **multiplicity** of *k*. Zeroes with a multiplicity of 1 are often called **simple** zeroes.
- For example, the polynomial $x^2 – 14x + 49$ will have one zero, x = 7, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as, $(x – 7)^2 = (x – 7)(x – 7)$
- Written this way the term (x – 7) shows up twice and each term gives the same zero, x = 7.
- Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

(F) Multiplicity ➔ Graphic Connection

Even Multiplicity **Odd Multiplicity**



(G) Examples - Applications

- ex 5. You have a sheet of paper 30 cm long by 20 cm wide. You cut out the 4 corners as squares and then fold the remaining four sides to make an open top box.
  - (a) Find the equation that represents the formula for the volume of the box.
  - (b) Find the volume if the squares cut out were each 2 cm by 2 cm.
  - (c) What are the dimensions of the squares that need to be removed if the volume is to be 1008 cm^3?

(G) Examples - Applications

- The volume of a rectangular-based prism is given by the formula $V(x) = x^3 – 5x^2 – 8x + 12$
  - (i) Express the height, width, depth of the prism in terms of *x*
  - (ii) State any restrictions for *x*. Justify your choice
  - (iii) what would be the dimensions on a box having a volume of 650 cubic units?
  - (iv) now use graphing technology to generate a reasonable graph for V(x). Justify your window/view settings

(G) Examples - Applications

- The equation $p(m) = 6m^5 – 15m^4 – 10m^3 + 30m^2 + 10$ relates the production level, p, in thousands of units as a function of the number of months of labour since October, m.
- Use graphing technology to graph the function and determine the following:
  - maximums and minimums. Interpret in context
  - Intervals of increase and decrease. Interpret
  - Explain why it might be realistic to restrict the domain. Explain and justify a domain restriction
  - Would $0 \leq m \leq 3$ be a realistic domain restriction?
- Find when the production level is 15,500 units (try this one algebraically as well)