## Lesson 21 - Review of Trigonometry

IB Math HL – Santowski



## BIG PICTURE

- The first of our keys ideas as we now start our Trig Functions & Analytical Trig Unit:
- (1) **How do we use current ideas to develop new ones**



## BIG PICTURE

- The first of our keys ideas as we now start our Trig Functions & Analytical Trig Unit:
- (1) **How do we use current ideas to develop new ones** ➔ We will use RIGHT TRIANGLES and CIRCLES to help develop new understandings



## BIG PICTURE

- The second of our keys ideas as we now start our Trig Functions & Analytical Trig Unit:
- (2) **What does a TRIANGLE have to do with SINE WAVES**



## BIG PICTURE

- The second of our keys ideas as we now start our Trig Functions & Analytical Trig Unit:
- (2) **What does a TRIANGLE have to do with SINE WAVES** ➔ How can we REALLY understand how the sine and cosine ratios from right triangles could ever be used to create function equations that are used to model periodic phenomenon



## Right Triangles

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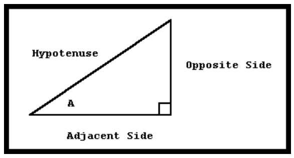
## (A) Review of Right Triangle Trig

- Trigonometry is the study and solution of Triangles. Solving a triangle means finding the value of each of its sides and angles. The following terminology and tactics will be important in the solving of triangles.

- Pythagorean Theorem ($a^2+b^2=c^2$). Only for right angle triangles
- Sine (sin), Cosecant (csc or 1/sin) ratios
- Cosine (cos), Secant (sec or 1/cos) ratios
- Tangent (tan), Cotangent (cot or 1/tan) ratios
- Right/Oblique triangle



## (A) Review of Right Triangle Trig




- In a right triangle, the primary trigonometric ratios (which relate pairs of sides in a ratio to a given reference angle) are as follows:

- sine A = opposite side/hypotenuse side & the cosecant A = cscA = h/o
- cosine A = adjacent side/hypotenuse side & the secant A = secA = h/a
- tangent A = adjacent side/opposite side & the cotangent A = cotA = a/o

- recall SOHCAHTOA as a way of remembering the trig. ratio and its corresponding sides



## (B) Review of Trig Ratios

- Evaluate and interpret:

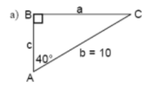
- (a) sin(32° )
- (b) cos(69° )
- (c) tan(10° )
- (d) csc(78° )
- (e) sec(13° )
- (f) cot(86° )

- Evaluate and interpret:

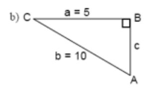
- (a) sin(x) = 0.4598
- (b) cos(x) = 0.7854
- (c) tan(x) = 1.432
- (d) csc(x) = 1.132
- (e) sec(x) = 1.125
- (f) cot(x) = 0.2768



## (C) Review of Trig Ratios and Triangles

- 


C = __________
a = __________
c = __________

- 


A = __________
C = __________
c = __________



## (B) Review of Trig Ratios

- If sin(x) = 2/3, determine the values of cos(x) & cot(x)
- If cos(x) = 5/13, determine the value of sin(x) + tan(x)
- If tan(x) = 5/8, determine the sum of sec(x) + 2cos(x)
- If tan(x) = 5/9, determine the value of $\sin^2(x) + \cos^2(x)$
- A right triangle with angle α = 30° has an adjacent side ***X*** units long. Determine the lengths of the hypotenuse and side opposite α.



# RADIAN MEASURE

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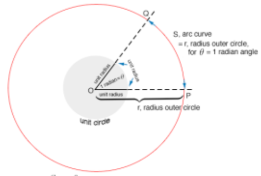
## (B) Radians

- We can measure angles in several ways - one of which is degrees
- Another way to measure an angle is by means of radians
- One definition to start with ➔ an arc is a distance along the curve of the circle ➔ that is, part of the circumference
- One radian is defined as the measure of the angle subtended at the center of a circle by an arc equal in length to the radius of the circle



## (B) Radians




If we rotate a terminal arm (OP) around a given angle, then the end of the arm (at point Q) moves along the circumference from P to Q

If the distance point P moves is equal in measure to the radius, then the angle that the terminal arm has rotated is defined as one radian

$$\frac{S}{|\vec{r}|} = \frac{\theta}{1}$$

or

$S = r\theta$, where

value of $\theta = 1.0$ rad or radian

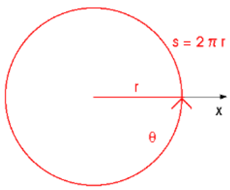
value of $|\vec{r}| = r$, radius of outer circle

If P moves along the circumference a distance twice that of the radius, then the angle subtended by the arc is 2 radians

So we come up with a formula of θ = arc length/radius = s/r



## (C) Converting between Degrees and Radians




If point B moves around the entire circle, it has revolved or rotated 360˚

Likewise, how far has the tip of the terminal arm traveled? One circumference or 2πr units.

So in terms of radians, the formula is θ = arc length/radius
θ = s/r = 2πr/r = 2π radians

- So then an angle of 360˚ = 2π radians
- or more easily, an angle of 180˚ = π radians



## (C) Converting from Degrees to Radians

- Our standard set of first quadrant angles include 0˚, 30˚, 45˚, 60˚, 90˚ and we now convert them to radians:
- We can set up equivalent ratios as:
- 30˚ =
- 45˚ =
- 60˚ =
- 90˚ =

- Convert the following angles from degree measure to radian measure:
- 21.6˚
- 138˚
- 72˚
- 293˚

## (D) Converting from Radians to Degrees

- Let's work with our second quadrant angles with our equivalent ratios:
- 2π/3 radians
- 3π/4 radians
- 5π/6 radians

- Convert the following angles from degree measure to radian measure:
- 4.2 rad
- 0.675 rad
- 18 rad
- 5.7 rad

## (E) Table of Equivalent Angles

- We can compare the measures of important angles in both units on the following table:

| 0 ˚ | 90˚ | 180˚ | 270˚ | 360˚ |
| --- | --- | --- | --- | --- |
| | | | | |

## (B) Review of Trig Ratios

- Evaluate and interpret:
- (a) sin(0.32)
- (b) cos(1.69)
- (c) tan(2.10)
- (d) csc(0.78)
- (e) sec(2.35)
- (f) cot(0.06)

- Evaluate and interpret:
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# Angles in Standard Position

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## QUIZ

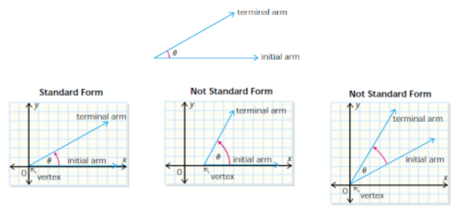
- Draw the following angles in standard position

- 70°
- 195°
- 140°
- 315°
- 870°
- -100°
- 4 radians



## (A) Angles in Standard Position

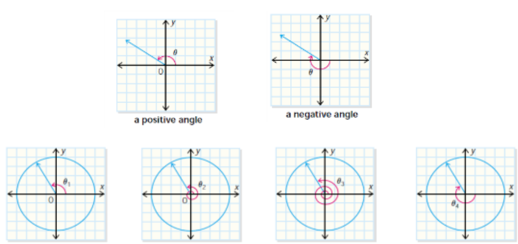
- Angles in standard position are defined as angles drawn in the Cartesian plane where the initial arm of the angle is on the *x* axis, the vertex is on the origin and the terminal arm is somewhere in one of the four quadrants on the Cartesian plane






## (A) Angles in Standard Position

- To form angles of various measure, the terminal arm is simply rotated through a given angle



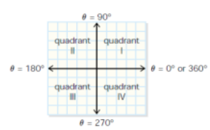



## (A) Angles in Standard Position

- We will divide our Cartesian plane into 4 quadrants, each of which are a multiple of 90 degree angles

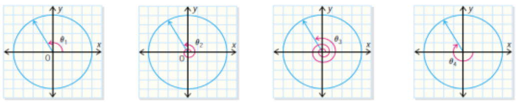
The $x$-$y$ plane is divided into four **quadrants** by the $x$- and $y$-axes. If $\theta$ is a positive angle, then the terminal arm lies in

- quadrant I when $0° < \theta < 90°$
- quadrant II when $90° < \theta < 180°$
- quadrant III when $180° < \theta < 270°$
- quadrant IV when $270° < \theta < 360°$



## (A) Coterminal Angles

- **Coterminal angles** share the same terminal arm and the same initial arm.
- As an example, here are four different angles with the same terminal arm and the same initial arm.

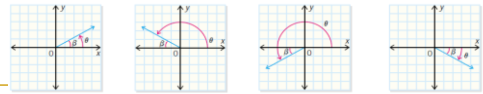


If $\theta_1 = 120°$, then $\theta_2 = 360° + 120° = 480°$ $\quad$ $\theta_3 = 720° + 120° = 840°$ $\quad$ $\theta_4 = -360° + 120° = -240°$



## (A) Principle Angles and Related Acute Angles

- The **principal angle** is the angle between 0° and 360° .
- The coterminal angles of 480° , 840° , and 240° all share the same principal angle of 120° .
- The **related acute angle** is the angle formed by the terminal arm of an angle in standard position and the *x*-axis.
- The related acute angle is always positive and lies between 0° and 90° .





## (B) Examples

- **Example 1**

  Determine the principal angle and the related acute angle for $\theta = -225°$.



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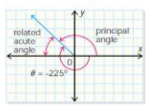
  Determine the principal angle and the related acute angle for $\theta = -225°$.

**Solution**

Sketch $\theta = -225°$ terminating in quadrant II. Label the principal angle and the related acute angle.




The principal angle is the smallest positive angle that is coterminal to $-225°$. In this case, $360° - 225° = 135°$. The related acute angle lies between the terminal arm and the $x$-axis. It is positive but less than 90°. In this case, $|-225° - (-180°)| = 45°$. Or, using the principal angle, $180° - 135° = 45°$.



## (B) Examples

- **Example 2**

  Determine the next two consecutive positive coterminal angles and the first negative coterminal angle for 43°.



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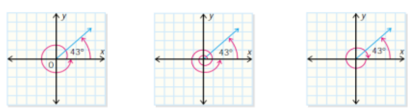
- **Example 2**

  Determine the next two consecutive positive coterminal angles and the first negative coterminal angle for 43°.

**Solution**

Sketch each situation, showing the principal angle of 43°.




**(a)** The first positive coterminal angle for 43° is $360° + 43° = 403°$.

**(b)** The second coterminal angle is $360° + 360° + 43° = 763°$.

**(c)** The first negative coterminal angle is $-360° + 43° = -317°$.

## (B) Examples

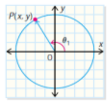
- For the given angles, determine:
- (a) the principle angle
- (b) the related acute angle (or reference angle)
- (c) the next 2 positive and negative co-terminal angles

- (i) $143°$
  (ii) $-132°$
  (iii) $419°$
  (iv) $-60°$
  (v) $4\ radians$
  (vi) $-\frac{17\pi}{12}$
  (vii) $\frac{7\pi}{6}$
  (viii) $-5.25\ radians$

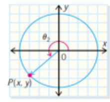


## (C) Ordered Pairs & Right Triangle Trig

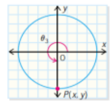
- To help discuss angles in a Cartesian plane, we will now introduce ordered pairs to place on the terminal arm of an angle




$90° < \theta_1 < 180°$
$\theta_1$ terminates in quadrant II.




$180° < \theta_2 < 270°$
$\theta_2$ terminates in quadrant III.

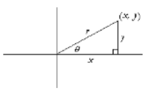



$P(x, y)$ lies in the negative $y$-axis.
$\theta_3 = 270°$



## (C) Ordered Pairs & Right Triangle Trig

- So to revisit our six trig ratios now in the context of the xy co-ordinate plane:
- We have our simple right triangle drawn in the first quadrant




$$\sin\theta = \frac{o}{h} = \frac{y}{r} \qquad \csc\theta = \frac{h}{o} = \frac{r}{y}$$
$$\cos\theta = \frac{a}{h} = \frac{x}{r} \qquad \sec\theta = \frac{h}{a} = \frac{r}{x}$$
$$\tan\theta = \frac{o}{a} = \frac{y}{x} \qquad \cot\theta = \frac{a}{o} = \frac{x}{y}$$



## (C) EXAMPLES

- Point *P* (-3, 4) is on the terminal arm of an angle, θ, in standard position.
- **(a)** Sketch the principal angle, θ and show the related acute/reference angle
- **(b)** Determine the values of all six trig ratios of θ.
- **(c)** Determine the value of the related acute angle to the nearest degree and to the nearest tenth of a radian.
- **(d)** What is the measure of θ to the nearest degree and to the nearest tenth of a radian?



## (C) Examples

- Point $P(-9, 4)$ is on the terminal arm of an angle in standard position.
  **(a)** Sketch the principal angle, $\theta$.
  **(b)** What is the measure of the related acute angle to the nearest degree?
  **(c)** What is the measure of $\theta$ to the nearest degree?

Point $P(-5, -3)$ is on the terminal arm of an angle, $\theta$, in standard position.
**(a)** Sketch the principal angle, $\theta$.
**(b)** What is the measure of the related acute angle to the nearest degree?
**(c)** What is the measure of $\theta$ to the nearest degree?
**(d)** What is the measure of the first negative coterminal angle?

Point $P(-5, -8)$ is on the terminal arm of an angle, $\theta$, in standard position. Determine all values of $\theta$ for $-540° \leq \theta \leq 270°$.



## (C) Examples

- Determine the angle that the line 2y + x = 6 makes with the positive x axis

# Working with Special Triangles

IB Math HL



## (A) Review – Special Triangles

- Review 45° - 45° - 90° triangle

- sin(45° ) = sin(π/4) =
- cos(45° ) = cos(π/4) =
- tan(45° ) = tan(π/4) =
- csc(45° ) = csc(π/4) =
- sec(45° ) = sec(π/4) =
- cot(45° ) = cot(π/4) =



## (A) Review – Special Triangles

- Review 30° - 60° - 90° triangle ➔ 30° ➔ π/6 rad

- sin(30° ) = sin(π/6) =
- cos(30° ) = cos(π/6) =
- tan(30° ) = cot(π/6) =
- csc(30° ) = csc(π/6) =
- sec(30° ) = sec(π/6) =
- cot(30° ) = cot(π/6) =

- Review 30° - 60° - 90° triangle ➔ 60° ➔ π/3 rad

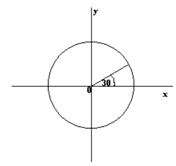
- sin(60° ) = sin(π/3) =
- cos(60° ) = cos(π/3) =
- tan(60° ) = tan(π/3) =
- csc(60° ) = csc(π/3) =
- sec(60° ) = sec(π/3) =
- cot(60° ) = cot(π/3) =



## (B) Trig Ratios of First Quadrant Angles

- We have already reviewed first quadrant angles in that we have discussed the sine and cosine (as well as other ratios) of 30° , 45° , and 60° angles

- What about the quadrantal angles of 0 ° and 90° ?

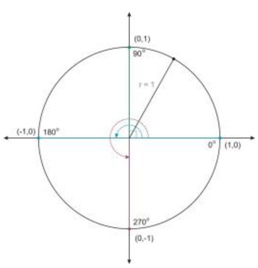





## (B) Trig Ratios of First Quadrant Angles – Quadrantal Angles

- Let's go back to the x,y,r definitions of sine and cosine ratios and use ordered pairs of angles whose terminal arms lie on the positive x axis (0° angle) and the positive y axis (90° angle)

- sin(0° ) =
- cos (0° ) =
- tan(0° ) =
- sin(90° ) = sin(π/2) =
- cos(90° ) = cos(π/2) =
- tan(90° ) = tan(π/2) =

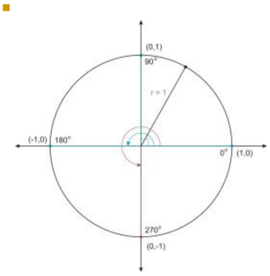





## (B) Trig Ratios of First Quadrant Angles – Quadrantal Angles

- Let's go back to the x,y,r definitions of sine and cosine ratios and use ordered pairs of angles whose terminal arms lie on the positive x axis (0° angle) and the positive y axis (90° angle)

- sin(0° ) = 0/1 = 0
- cos (0° ) = 1/1 = 1
- tan(0° ) = 0/1 = 0
- sin(90° ) = sin(π/2) =1/1 = 1
- cos(90° ) = cos(π/2) =0/1 = 0
- tan(90° ) = tan(π/2) =1/0 = undefined



## (B) Trig Ratios of First Quadrant Angles - Summary

|  | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| sin $\theta$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| cos $\theta$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 |
| tan $\theta$ | 0 | $\frac{\sqrt{3}}{3}$ | 1 | $\sqrt{3}$ | $\pm\infty$ |

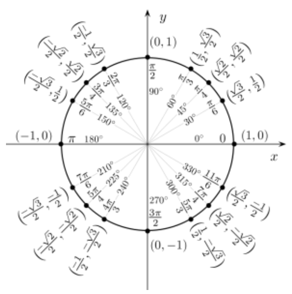


## (G) Summary – As a "Unit Circle"

- The Unit Circle is a tool used in understanding sines and cosines of angles found in right triangles.
- It is so named because its radius is exactly one unit in length, usually just called "one".
- The circle's center is at the origin, and its circumference comprises the set of all points that are exactly one unit from the origin while lying in the plane.



## (G) Summary – As a "Unit Circle"






## (H) EXAMPLES

- Simplify or solve

$(a)\ \sin 30°\cos 30° - \tan 30°$

$(b)\ \sin 45°\sin 30° - (\tan 60°)^2$

$(c)\ \dfrac{\sin 150°}{\sec 210°} - \csc(-330°)$

$(b)\ \sin(\theta) = -\dfrac{1}{2}$

$(c)\ 2\cos(\theta) = 1$

$(d)\ \sqrt{3}\tan(\theta) = 1$



## (H) EXAMPLES

- Simplify the following:

(a) $\sin^2\left(\frac{2\pi}{3}\right) + \cos^2\left(\frac{2\pi}{3}\right) =$

(b) $\dfrac{\sin(225°)}{\cos(225°)}$ compared to $\tan(225°)$

(c) $2\sin\left(-\frac{\pi}{6}\right)\cos\left(-\frac{\pi}{6}\right)$ compared to $\sin\left(-\frac{\pi}{3}\right)$