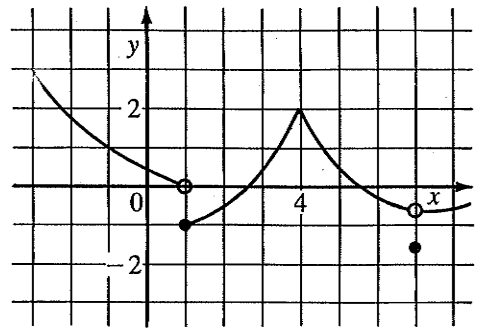
1.8 REVIEW EXERCISE

1. Use the given graph of f to state the value of the limit, if it exists.



(a) $\lim_{x \to -1} f(x)$ (b) $\lim_{x \to 1^-} f(x)$

(c) $\lim_{x \to 1^+} f(x)$ (d) $\lim_{x \to 1} f(x)$

(e) $\lim_{x \to -3^+} f(x)$ (f) $\lim_{x \to 4^-} f(x)$

(g) $\lim_{x \to 4^+} f(x)$ (h) $\lim_{x \to 4} f(x)$

2. State whether the function f, whose graph is shown in Question 1, is continuous or discontinuous at the following numbers.

(a) 1 (b) 4 (c) 7

3. Find the following limits.

(a) $\lim_{x \to 2} (3x^3 + 7x - 16)$ (b) $\lim_{x \to -1} \dfrac{2x + 3}{3x + 2}$

(c) $\lim_{x \to 2} \dfrac{x^2 - 2x - 8}{x^2 - 7x + 12}$ (d) $\lim_{x \to 4} \dfrac{x^2 - 2x - 8}{x^2 - 7x + 12}$

(e) $\lim_{x \to 5} \sqrt{\dfrac{x^2 - 25}{x - 5}}$ (f) $\lim_{x \to 4} \dfrac{x - 4}{x^3 - 64}$

(g) $\lim_{t \to 0} \dfrac{\sqrt{2 + t} - \sqrt{2}}{t}$ (h) $\lim_{h \to 0} \dfrac{(-3 + h)^2 - 9}{h}$

4. Find the following limits, or state that they do not exist.

(a) $\lim_{x \to -1} \dfrac{x - 6}{(x + 1)^3}$ (b) $\lim_{x \to 1} \dfrac{x^2 - 1}{x^2 - 7x + 6}$

(c) $\lim_{h \to 0} \dfrac{\dfrac{4}{2 + h} - 2}{h}$ (d) $\lim_{y \to 2} \dfrac{y^4 - 16}{y^4 + 2y^3 - y^2 - 2y}$

(e) $\lim_{t \to -2^+} \sqrt[4]{8 + t^3}$ (f) $\lim_{x \to 1^+} \dfrac{|x - 1|}{x - 1}$

(g) $\lim_{x \to 1^-} \dfrac{|x - 1|}{x - 1}$ (h) $\lim_{x \to 1} \dfrac{|x - 1|}{x - 1}$

5. Let
$$f(x) = \begin{cases} -1 - x & \text{if } x < -1 \\ x^2 & \text{if } x \geqslant -1 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x \to -1^-} f(x)$ (ii) $\lim_{x \to -1^+} f(x)$ (iii) $\lim_{x \to -1} f(x)$

(b) Sketch the graph of f.

6. Let
$$g(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } 0 \leqslant x \leqslant 1 \\ 1 + 2x - x^2 & \text{if } x > 1 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x \to 0^-} g(x)$ (ii) $\lim_{x \to 0^+} g(x)$ (iii) $\lim_{x \to 0} g(x)$

(iv) $\lim_{x \to 1^-} g(x)$ (v) $\lim_{x \to 1^+} g(x)$ (vi) $\lim_{x \to 1} g(x)$

(b) Sketch the graph of g.

(c) Where is g discontinuous?

7. A daytime coin-paid phone call from Toronto to Montreal costs \$1.95 for the first minute and \$0.45 for each additional minute (or part of a minute). Draw the graph of the cost C (in dollars) of the phone call as a function of the time t (in minutes). For what values of t does this function have discontinuities?

8. The point $P(1, -2)$ lies on the curve $y = x^3 - 3x$.

(a) If Q is the point $(x, x^3 - 3x)$, find the slope of the secant line PQ for the following values of x:

(i) 2 (ii) 1.5 (iii) 1.1 (iv) 1.01

(b) Find the slope of the tangent line to the curve at P.

(c) Find an equation of the tangent line to the curve at P.

(d) Graph the curve and the tangent line.

9. Find the equation of the tangent line to the curve $y = x^4$ at the point $(-1, 1)$.

10. If a stone is dropped off a 200 m high cliff, then its height after t seconds, and before it hits the ground, is $h = 200 - 4.9t^2$.

(a) Find the average velocity of the stone for the following time periods.

(i) $1 \leqslant t \leqslant 2$ (ii) $1 \leqslant t \leqslant 1.1$

(b) Find the instantaneous velocity when $t = 1$.

11. A spherical balloon is being inflated. Find the rate of change of the surface area of the balloon with respect to the radius when the radius is 10 cm. (Use the formula $S = 4\pi r^2$, where r is the radius of a sphere and S is the surface area.)

12. Find the following limits or state that the limit does not exist.

(a) $\lim_{n \to \infty} \left(2 - \frac{1}{n} + \frac{3}{n^2}\right)$ (b) $\lim_{n \to \infty} \frac{1 + 2n}{1 - 3n}$

(c) $\lim_{n \to \infty} (1.1)^n$ (d) $\lim_{n \to \infty} \frac{3^n}{5^n}$

1.9 CHAPTER 1 TEST

1. Find the following limits.

(a) $\lim_{x \to 2} \sqrt{\dfrac{x^2 + 5}{x - 1}}$

(b) $\lim_{x \to -1} \dfrac{x + 1}{x^2 - 4x - 5}$

(c) $\lim_{x \to 1} \dfrac{\dfrac{1}{\sqrt{x}} - 1}{x - 1}$

2. The points $P(2, -1)$ and $Q(3, -4)$ lie on the parabola $y = -x^2 + 2x - 1$.

(a) Find the slope of the secant line PQ.

(b) Find the slope of the tangent line to the parabola at P.

(c) Find the equation of the tangent line at P.

(d) Graph the parabola, the secant line, and the tangent line.

3. Let
$$f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x - 1 & \text{if } x > 0 \end{cases}$$

(a) Find the following limits if they exist.

(i) $\lim_{x \to 0^-} f(x)$ (ii) $\lim_{x \to 0^+} f(x)$ (iii) $\lim_{x \to 0} f(x)$

(b) Sketch the graph of f.

(c) Where is f discontinuous?

4. The displacement in metres of a particle moving in a straight line is given by $s = 5t^2 - 6t + 14$, where t is measured in seconds.

(a) Find the average velocity over the time interval $2 \leqslant t \leqslant 3$.

(b) Find the instantaneous velocity when $t = 2$.

5. Evaluate $\lim_{n \to \infty} \left(\dfrac{1}{8^n} + \dfrac{6n - 2}{2n - 3}\right)$.

6. Find the sum of the series
$$12 - 9 + \frac{27}{4} - \frac{81}{16} + \cdots$$

2.9 REVIEW EXERCISE

1. Find $f'(x)$ from first principles, that is, directly from the definition of a derivative.

(a) $f(x) = 1 - 2x + 3x^2$ (b) $f(x) = x^3 + 4x$

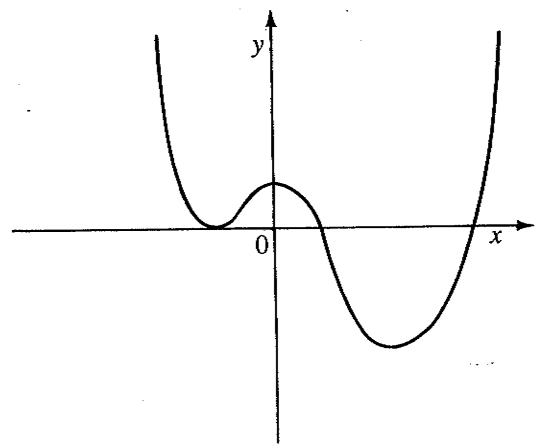
(c) $f(x) = \dfrac{x}{1 - x}$ (d) $f(x) = \sqrt{2x + 1}$

2. The limit

$$\lim_{h \to 0} \frac{(1 + h)^4 - 1}{h}$$

is equal to $f'(a)$ for some function f and some number a. State the value of a and give a formula for the function f.

3. Use the given graph of f to sketch the graph of f'.



4. Differentiate the given functions.

(a) $y = 12x^3 + 8x - 1$ (b) $y = 2x^{\pi + 1}$

(c) $y = 2x - \dfrac{3}{x}$ (d) $y = \sqrt[5]{x^6}$

(e) $y = \sqrt{x}(5 - \sqrt{x})$ (f) $y = \dfrac{x^2 - 2x}{\sqrt{x}}$

(g) $y = \dfrac{2x - 1}{1 + 3x}$ (h) $y = (2x^3 - 1)^7$

(i) $f(x) = (x^2 + x)\sqrt{1 - x^2}$ (j) $g(x) = \dfrac{3x^2 + 1}{2 - x}$

(k) $h(x) = \dfrac{1}{\sqrt[3]{2x^4 - 1}}$ (l) $F(x) = (x^4 + 1)^3(1 - 2x)$

(m) $f(t) = \dfrac{t}{\sqrt{1 + 2t}}$ (n) $g(t) = \left(\dfrac{t + 1}{t + 2}\right)^4$

(o) $R(u) = \sqrt[4]{u + 1} - \dfrac{2}{u^2}$ (p) $S(v) = \sqrt{v - (v^2 - 8)^5}$

(q) $M(z) = \sqrt{\dfrac{1 + z}{1 + z^2}}$ (r) $F(y) = \dfrac{1}{2 + \dfrac{3}{y}}$

5. Find f' and state the domains of f and f'.

(a) $f(x) = \dfrac{2x - 1}{x^2 - 5}$ (b) $f(x) = \sqrt{x^2 - x - 6}$

6. Find $\left[\dfrac{dy}{dx}\right]_{x=1}$ if $y = u^2 - u^3 + 2u^4$ and $u = \dfrac{x}{2x - 1}$.

7. Find $\dfrac{dy}{dx}$.

(a) $x^4 + y^4 = 1$ (b) $x^2 - x^2y + y^2 = 1$

(c) $2x^2y^2 = x^3 + y^3$ (d) $y\sqrt{x - 1} + x\sqrt{y - 1} = xy$

8. Find y''.

(a) $y = 4x^5 - \frac{1}{2}x^4 + 3x^2$ (b) $y = \sqrt{3x + 1}$

(c) $y = \dfrac{t - 1}{t + 1}$ (d) $x^2 + y^2 = 16$

9. Find the equation of the tangent line to the curve at the given point.

(a) $y = x^2 - 2x + 5$, $(-1, 8)$ (b) $y = \dfrac{2}{1 - x}$, $(2, -2)$

(c) $y = \dfrac{1}{\sqrt{x^5}}$, $\left(2, \dfrac{1}{4\sqrt{2}}\right)$ (d) $y = x\sqrt{x^2 + 5}$, $(-2, -6)$

(e) $(x - 1)^2 + (y + 2)^2 = 25$, $(-2, 2)$

(f) $x^3 + y^3 = 9xy$, $(2, 4)$

10. If a ball is dropped from the top of the CN Tower, 550 m above the ground, then its height in metres after t seconds is $h = 550 - 5t^2$. Find the velocity of the ball after 1 s, 2 s, and 5 s.

11. Find the point on the parabola $y = 2x^2 - 3x + 6$ where the tangent line is parallel to the line $7x + y = 1$.

12. Find the points on the curve $y = \dfrac{1}{2x - 1}$ where the tangent line is perpendicular to the line $x - 2y = 1$.

13. Find the equations of both lines that pass through the point $(2, -3)$ and are tangent to the parabola $y = x^2 + x$.

14. Suppose $f(3) = 4$, $f'(3) = -1$, $f'(6) = 5$, $g(3) = 6$, and $g'(3) = 2$. Find

(a) $(fg)'(3)$ (b) $\left(\dfrac{f}{g}\right)'(3)$ (c) $(f \circ g)'(3)$

15. If g is a differentiable function, find expressions for f' in terms of g'.

(a) $f(x) = x^2 g(x)$ (b) $f(x) = \dfrac{g(x)}{\sqrt{x}}$

(c) $f(x) = g\left(\dfrac{1}{x}\right)$ (d) $f(x) = \sqrt{g(\sqrt{x})}$

16. If g is a differentiable function and $f(x) = g(g(x))$, find an expression for $f''(x)$.

17. Let

$$f(x) = \begin{cases} 2x - x^2 & \text{if } x \leq 0 \\ 2x & \text{if } 0 < x \leq 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$$

(a) Where is f not differentiable?

(b) Find an expression for $f'(x)$ and sketch the graphs of f and f'.

PROBLEMS PLUS

Draw a diagram to show that there are two lines tangent to both of the parabolas $y = -x^2$ and $y = 4 + x^2$. Find the coordinates of the four points at which these tangents touch the parabolas.

erms of

f and f'.

ent to
d the
touch

2.10 CHAPTER 2 TEST

1. (a) Give the definition of the derivative $f'(x)$ as a limit.
   (b) Use your definition in part (a) to find the derivatives of the following functions:

   (i) $f(x) = x^2 - 7x + 4$ (ii) $f(x) = \dfrac{1}{2x+1}$

2. Find each derivative.
   (a) $f(x) = \sqrt[3]{x^2}$
   (b) $f(x) = \dfrac{x^2+3}{2x-1}$
   (c) $f(x) = (x^2 - 1)^4(2x + 1)^3$
   (d) $f(x) = (x + \sqrt{x^4 - 2x + 1})^7$
3. A curve is given by the equation $3xy = x^3 + y^3$.
   (a) Find $\dfrac{dy}{dx}$.
   (b) Find the equation of the tangent line to the curve at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.
4. Find y''' if $y = \dfrac{1}{(3-2x)^2}$.
5. Find the point on the curve $y = \sqrt{2x-1}$ where the tangent line is parallel to the line $x - 3y = 16$.
6. If f is a differentiable function, find expressions for the derivatives of the following functions.
   (a) $g(x) = f(x^6)$
   (b) $h(x) = [f(x)]^6$
   (c) $F(x) = \dfrac{x^2}{f(x)}$