AP* Calculus Review

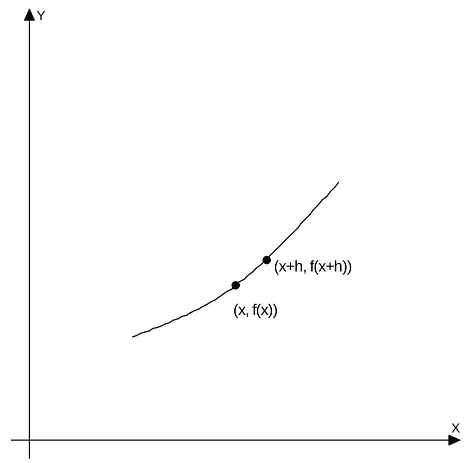
Limits, Continuity, and the Definition of the Derivative

Teacher Packet

DEFINITION **Derivative of a Function**

The **derivative** of the function f with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



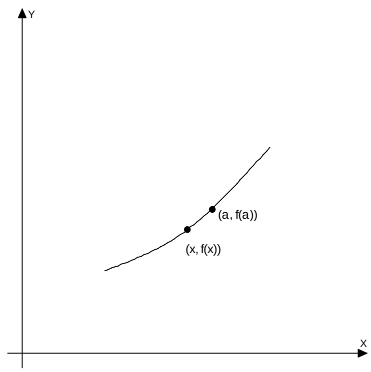
provided the limit exists.

You will want to recognize this formula (a slope) and know that you need to take the derivative of $f(x)$ when you are asked to find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

DEFINITION (ALTERNATE**)** **Derivative at a Point**

The **derivative** of the function ***f*** **at the point** $\boldsymbol{x = a}$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



provided the limit exists.

This is the slope of a segment connecting two points that are very close together.

DEFINITION **Continuity**

A function f is continuous at a number a if

1) $f(a)$ is defined (a is in the domain of f)

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

A function is continuous at an *x* if the function has a value at that *x*, the function has a limit at that *x*, and the value and the limit are the same.

Example:

Given $f(x) = \begin{cases} x^2 + 3, & x \le 2 \\ 3x + 2, & x > 2 \end{cases}$

Is the function continuous at $x = 2$?

$f(x) = 7$

$\lim_{x \to 2^-} f(x) = 7$, but the $\lim_{x \to 2^+} f(x) = 8$

The function does not have a limit as $x \to 2$, therefore the function is not continuous at $x = 2$.

Limits as *x* approaches ∞

For rational functions, examine the *x* with the largest exponent, numerator and denominator. The *x* with the largest exponent will carry the weight of the function.

If the *x* with the largest exponent is in the denominator, the denominator is growing faster as $x \to \infty$. Therefore, the limit is 0.

$$\lim_{x \to \infty} \frac{3+x}{x^4 - 3x + 7} = 0$$

If the *x* with the largest exponent is in the numerator, the numerator is growing faster as $x \to \infty$. The function behaves like the resulting function when you divide the *x* with the largest exponent in the numerator by the *x* with the largest exponent in the denominator.

$$\lim_{x \to \infty} \frac{3+x^5}{x^2 - 3x + 7} = \infty$$

This function has end behavior like $x^3 \left(\frac{x^5}{x^2}\right)$. The function does not reach a limit, but to say the limit equals infinity gives a very good picture of the behavior.

If the *x* with the largest exponent is the same, numerator and denominator, the limit is the coefficients of the two *x's* with that largest exponent.

$\lim_{x \to \infty} \frac{3+4x^5}{7x^5 - 3x + 7} = \frac{4}{7}$. As $x \to \infty$, those x^5 terms are like gymnasiums full of sand. The few grains of sand in the rest of the function do not greatly affect the behavior of the function as $x \to \infty$.

LIMITS

$$\lim_{x \to c} f(x) = L$$

The limit of *f* of *x* as *x* approaches *c* equals *L*.

As *x* gets closer and closer to some number *c* (but does not equal *c*), the value of the function gets closer and closer (and may equal) some value *L*.

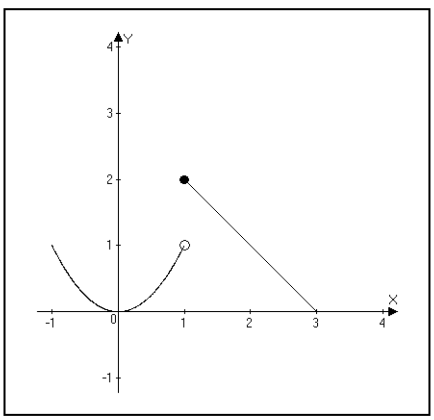
One-sided Limits

$$\lim_{x \to c^-} f(x) = L$$

The limit of *f* of *x* as *x* approaches *c* from the left equals *L*.

$$\lim_{x \to c^+} f(x) = L$$

The limit of *f* of *x* as *x* approaches *c* from the right equals *L*.



Using the graph above, evaluate the following:

$\lim_{x \to 1^-} f(x) =$ $\lim_{x \to 1^+} f(x) =$ $\lim_{x \to 1} f(x) =$

Practice Problems

Limit as *x* approaches infinity

1. $\lim_{x\to\infty}\left(\dfrac{3x-7}{5x^4-8x+12}\right)=$

2. $\lim_{x\to\infty}\left(\dfrac{3x^4-2}{5x^4-2x+1}\right)=$

3. $\lim_{x\to\infty}\left(\dfrac{x^6-2}{10x^4-9x+8}\right)=$

4. $\lim_{x\to\infty}\left(\dfrac{7x^4-2}{5-2x^3-14x^4}\right)=$

5. $\lim_{x\to\infty}\left(\dfrac{\sin x}{e^x}\right)=$

6. $\lim_{x\to-\infty}\left(\dfrac{\sqrt{x^2-9}}{2x-3}\right)=$

7. $\lim_{x\to\infty}\left(\dfrac{\sqrt{x^2-9}}{2x-3}\right)=$

Practice Problems

Limit as *x* approaches a number

8. $\lim_{x \to 2} \left(x^3 - x + 1 \right)$

9. $\lim_{x \to 2} \left(\dfrac{x^2 - 4}{x - 2} \right) =$

10. $\lim_{x \to 2^-} \left(\dfrac{3}{x - 2} \right) =$

11. $\lim_{x \to 2^+} \left(\dfrac{3}{x - 2} \right) =$

12. $\lim_{x \to 2} \left(\dfrac{3}{x - 2} \right) =$

13. $\lim_{x \to 2^+} \left(\dfrac{3}{2 - x} \right) =$

14. $\lim_{x \to \frac{\pi}{4}} \left(\dfrac{\sin x}{x} \right) =$

15. $\lim_{x \to \frac{\pi}{4}} \left(\dfrac{\tan x}{x} \right) =$

1. What is $\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$?

   (A) $\sin x$ (B) $\cos x$ (C) $-\sin x$

   (D) $-\cos x$ (E) The limit does not exist

2. $\lim_{\Delta x \to 0} \frac{\cos\left(\frac{\pi}{3} + \Delta x\right) - \cos\left(\frac{\pi}{3}\right)}{\Delta x} =$

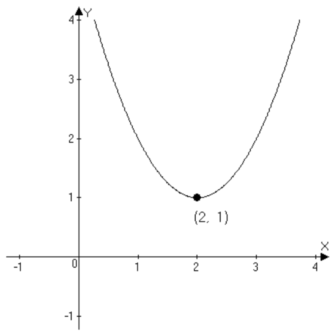
   (A) $-\frac{\sqrt{3}}{2}$ (B) $-\frac{1}{2}$ (C) 0

   (D) $\frac{1}{2}$ (E) $\frac{\sqrt{3}}{2}$

3. $\lim_{h \to 0} \frac{(x+h)^3 - (x^3)}{h} =$

   (A) $-x^3$ (B) $-3x^2$ (C) $3x^2$

   (D) x^3 (E) The limit does not exist



4. The graph of $y = f(x)$ is shown above. $\lim_{x \to 2}\left(\left(f(x)^3\right) - 3f(x) + 7\right) =$

(A) 1 (B) 5 (C) 7 (D) 9 (E) Does not exist

5. If $f(x) = \begin{cases} \dfrac{x^2 - 3x - 4}{x+1}, & x \neq -1 \\ 2, & x = -1 \end{cases}$, what is $\lim_{x \to -1} f(x)$?

(A) -5 (B) 0 (C) 2 (D) 3 (E) Does not exist

6. $\lim_{x \to \infty}\left(\dfrac{2x^6 - 5x^3 + 10}{20 - 4x^2 - x^6}\right) =$

(A) -2 (B) $-\dfrac{1}{2}$ (C) $\dfrac{1}{2}$ (D) 2 (E) Does not exist

7. $\lim_{x \to \infty}\left(\dfrac{2x^5 - 5x^3 + 10}{20 - 4x^2 - x^6}\right) =$

(A) -2 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$ (E) 2

8. $\lim_{x \to \infty} \left(1 + e^{\frac{1}{2} + \frac{1}{x}}\right) =$

(A) $-\infty$ (B) 0 (C) $e^{\frac{1}{2}}$

(D) $1 + e^{\frac{1}{2}}$ (E) ∞

9. $\lim_{x \to 3^+} \frac{5}{3 - x} =$

(A) $-\infty$ (B) -5 (C) 0

(D) $\frac{5}{3}$ (E) ∞

10. If $\lim_{x \to \infty} \left(\frac{5n^3}{20 - 3n - kn^3} \right) = \frac{1}{2}$, then $k =$

(A) -10 (B) -4 (C) $\frac{1}{4}$ (D) 4 (E) 10

11. Which of the following is/are true about the function g if $g(x) = \frac{(x-2)^2}{x^2 + x - 6}$?

I. g is continuous at $x = 2$
II. The graph of g has a vertical asymptote at $x = -3$
III. The graph of g has a horizontal asymptote at $y = 0$

(A) I only (B) II only (C) III only (D) I and II only (E) II and III only

12. $f(x) = \begin{cases} \sin x, & x < \frac{\pi}{4} \\ \cos x, & x > \frac{\pi}{4} \\ \tan x, & x = \frac{\pi}{4} \end{cases}$

What is $\lim\limits_{x \to \frac{\pi}{4}} f(x)$?

(A) $-\infty$ (B) 0 (C) 1 (D) $\frac{\sqrt{2}}{2}$ (E) ∞

13. $\lim\limits_{x \to a} \left(\frac{\sqrt{x} - \sqrt{a}}{x - a} \right) =$

(A) $\frac{1}{2\sqrt{a}}$ (B) $\frac{1}{\sqrt{a}}$ (C) $\sqrt{a}$ (D) $2\sqrt{a}$ (E) Does not exist

14. $\lim\limits_{x \to 0^+} \frac{\ln 2x}{2x} =$

(A) $-\infty$ (B) -1 (C) 0 (D) 1 (E) ∞

15. At $x = 4$, the function given by $h(x) = \begin{cases} x^2, & x \le 4 \\ 4x, & x > 4 \end{cases}$ is

(A) Undefined
(B) Continuous but not differentiable
(C) Differentiable but not continuous
(D) Neither continuous nor differentiable
(E) Both continuous and differentiable

Free Response 1

Let h be the function defined by the following:

$$h(x) = \begin{cases} |x-1| + 3, & 1 \le x \le 2 \\ ax^2 - bx, & x > 2 \end{cases}$$

a and b are constants.

(a) If $a = -1$ and $b = -4$, is $h(x)$ continuous for all x in $[1, \ \infty]$? Justify your answer.

(b) Describe all values of a and b such that h is a continuous function over the interval $[1, \ \infty]$.

(c) The function h will be continuous and differentiable over the interval $[1, \ \infty]$ for which values of a and b?

Free Response 2 (No calculator)

Given the function $f(x) = \dfrac{x^3 + 2x^2 - 3x}{3x^2 + 3x - 6}$.

(a) What are the zeros of $f(x)$?

(b) What are the vertical asymptotes of $f(x)$?

(c) The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?

(d) What is $\lim\limits_{x \to \infty} f(x)$? What is $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)}$?

Key

Page 5: 1, 2, does not exist

Practice Problems:

1. 0
2. $\frac{3}{5}$
3. ∞
4. $-\frac{1}{2}$
5. 0
6. $-\frac{1}{2}$
7. $\frac{1}{2}$
8. 7
9. 4
10. $-\infty$
11. ∞
12. does not exist
13. $-\infty$
14. $\frac{2\sqrt{2}}{\pi}$
15. $\frac{4}{\pi}$

Multiple Choice Questions:

1. B
2. A
3. C
4. B
5. A
6. A
7. C
8. D
9. A
10. A
11. B
12. D
13. A
14. A
15. B

Free Response 1

Let h be the function defined by the following:

$$h(x)=\begin{cases}|x-1|+3, & 1\le x\le 2\\ ax^2-bx, & x>2\end{cases}$$

a and b are constants.

(a) If $a=-1$ and $b=-4$, is $h(x)$ continuous for all x in $[1,\ \infty]$? Justify your answer.

(b) Describe all values of a and b such that h is a continuous function over the interval $[1,\ \infty]$.

(c) The function h will be continuous and differentiable over the interval $[1,\ \infty]$ for which values of a and b?

(a) $\lvert 2-1\rvert+3=4a-2b$ $4=4a-2b$ $4=4(-1)-2(-4)$ $4=4$	1 pt equation with substitutions
$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=4$ $f(2)=4$	1 pt limits equal; 1 pt value
The function, $h(x)$, is continuous for all x, given the a and b, because the function has a limit as x approaches 2, the function has a value as x approaches 2, and the limit is equal to the value.	
(b) $4a-2b=4$ $4a=4+2b$ $a=1+\frac{1}{2}b$	2 pts for finding a in terms of b
The function is continuous for all $a=1+\frac{1}{2}b$.	
(c) $4=4a-2b$ $1=4a-b$	1 pt for continuity equation 1 pt for differentiability equation

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$-b = 3$	
$b = -3$	1 pt for *a*; 1 pt for *b*
$a = -\frac{1}{2}$	
The function, *h(x)*, is continuous and differentiable when $a = -\frac{1}{2}$, $b = -3$	

Free Response 2 (No calculator)

Given the function $f(x)=\dfrac{x^3+2x^2-3x}{3x^2+3x-6}$.

(a) What are the zeros of $f(x)$?

(b) What are the vertical asymptotes of $f(x)$?

(c) The end behavior model of $f(x)$ is the function $g(x)$. What is $g(x)$?

(d) What is $\lim_{x\to\infty} f(x)$? What is $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$?

(a) The zeros of the function, $f(x)$, occur at $x=-3,\ 0,\ 1$	3 pts, 1 for each zero
(b) There is a vertical asymptote at $x=-2$	1 pt for the vertical asymptote
(c) $g(x)=\dfrac{1}{3}x$	2 pts for $g(x)$
(d) $\lim_{x\to\infty} f(x)=\infty$	1 pt $\lim_{x\to\infty} f(x)$
$\lim_{x\to\infty}\dfrac{f(x)}{g(x)}=1$	2 pts for $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}$